AN EXPLICIT EXPRESSION FOR k-NEGATIONS

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Abstract

We give a new proof for the existence and the uniqueness of k-negations. With the aid of a generalised dyadic representation system is possible to give an explicit expression for k-negations and it is possible to showing that there exists a set of measure 1 where k-negations have derivatives with zero values.

Keywords: Negation, k-negation, generalised dyadic system, singular function

1 Introduction

Negation functions are very well known objects from Set Theory and Fuzzy Logic: they are mappings $n : [0,1] \to [0,1]$ satisfying *a)* $n(0) = 1$, $n(1) = 0$; *b)* n is not increasing. In case n satisfies *c)* $n^2(x) = x$, we say that n is a strong negation.

In a classic paper by Trillas [14], we can find the following characterization for strong negations:

Theorem 1 *The function n is a strong negation if, and only if, there exists an increasing bijection $f : [0,1] \to [0,1]$, verifying $n(x) = f^{-1}(1 - f(x))$.*

A particular case of great interest is that of negations which are related with the duality of aggregation functions and the idempotentity of some binary functions. They are called k-negations and are denoted by n_k. In fact, they are strong negations satisfying this additional property (for $k \in]0,1[$):

$$n_k(kx) = k + (1-k)\, n_k(x),$$

and they have been studied in [5], [6], and [7], among others. The main fact is that for each k, there exists one, and only one, n_k.

The solution for the system of functional equations

$$\begin{cases} h(\frac{x}{2}) = kh(x) \\ h(\frac{1+x}{2}) = k + (1-k)h(x) \end{cases}$$

with x in $[0,1]$, is a function which inverse satisfies the relation of Trillas for k-negations. It has been deeply studied as we can see in [2], [3], [4], [9], [10], [11], [12], and [13]. We will denote it with the capital letter R.

The function R is (for $k \neq 1/2$) singular in the sense of Measure Theory (i.e., it is a continuous and monotone function having null derivatives on a set of measure 1).

The relation

$$n_k(x) = R\left(1 - R^{-1}(x)\right)$$

does not yields to an explicit expression for $n_k(x)$. In this sense Fraile et al. [5], gave a recurrence expression on a dense and denumerable subset of $[0,1]$ which is generalized in [8].

The target of this paper is double: on the one hand, we give a explicit expression for k-negations, and, on the other hand, we show this new property for n_k: it has null derivatives on a set of measure 1 (i.e., the same property that was true for R), and moreover, this is the only value that the derivatives can get.

The main tools we will use to obtain these results are a generalised dyadic system developed by the authors in [1], and the system of functional equations characterizing k-negations.

2 The existence of k-negations

Lemma 2 *The system of functional equations*

$$\begin{cases} h(kx) = kh(x) \\ h(k + (1-k)x) = k + (1-k)h(x) \end{cases}$$

has one, and only one, solution in the set $\mathcal{B}([0,1], \mathbb{R})$ of the real bounded functions defined on the unit interval; and it is the identity map.

Proof. Clearly, the identity map satisfies the system of equations.

For short, let us denote $\mathcal{B}([0,1])$ the Banach space of real bounded functions defined on the unit interval equipped with the sup-norm, and the functional operator introduced by the formula

$$F : \mathcal{B}([0,1]) \longrightarrow \mathcal{B}([0,1])\,; \qquad g \to F(g),$$

where $F(g)$ acts in the following way:

$$F(g)(y) := \begin{cases} kg\left(\frac{y}{k}\right), & 0 \le y \le k \\ k + (1-k)g\left(\frac{y-k}{1-k}\right), & k < y \le 1. \end{cases}$$

Hence, F is contractive of ratio $b := \max\{k, 1-k\}$. For this, let us denote $h_i := F(g_i)$ for given g_1 and g_2 in $\mathcal{B}([0,1])$. Easy computations show that

$h_2(y) - h_1(y) = k\left[g_2\left(\frac{y}{k}\right) - g_1\left(\frac{y}{k}\right)\right]$ if $0 \le y \le k$, and

$h_2(y) - h_1(y) = (1-k)\left[g_2\left(\frac{y-k}{1-k}\right) - g_1\left(\frac{y-k}{1-k}\right)\right]$ if $k \le y \le 1$,

implies $\|h_2 - h_1\| \le b\,\|g_2 - g_1\|$.

The contractive mapping theorem of Banach gives that there exist one and only one fix point for F: the desired uniqueness of h is just fulfilled, and the proof is finished. ■

Lemma 3 *One element h in $\mathcal{B}([0,1])$ is a k-negation if, and only if, it satisfies the equations*

$$\begin{cases} h(kx) = k + (1-k)h(x) \\ h(k + (1-k)x) = kh(x) \end{cases} \qquad (a)$$

Proof. A k-negation satisfies

$$\begin{cases} n_k(kx) = k + (1-k)n_k(x) \\ n_k^2(x) = x \end{cases}.$$

Because $n_k(n_k(kx)) = kx$, we can write

$n_k\,(k + (1-k)n_k\,(x)) = kx$; and doing $n_k\,(x) = y$ we have $n_k(k + (1-k)y) = kn_k^{-1}(y)$.

But $n_k^{-1} = n_k$, and hence, we obtain the second of the equations in (a).

For the reverse implication it is enough to show that $n_k^2(x) = x$. From (a), we obtain

$$\begin{cases} n_k^2(kx) = n_k(k + (1-k)n_k(x)) = kn_k^2(x) \\ n_k^2(k + (1-k)x) = n_k(kn_k(x)) = k + (1-k)n_k^2(x) \end{cases}$$

These equalities and the lemma above ensure that $n_k^2(x) = x$. Moreover, one can check that n_k is decreasing. ■

3 An explicit expression for k-negations

3.1 Generalised dyadic number system

In this section, we refer to [1, \$.3] for a wider study in detail.

The authors introduce a new representation system for numbers in $]0,1[$ via series expansion combining the numbers k and $1-k$ ($k \in]0,1[$); precisely, in the form $x = \sum_{n=1}^{+\infty} (1-k)^n\, k^{m_n}$. This situation is unique, very similar to that of dyadic expansions, but a denumerable set of numbers for which there are exactly two representations: one finite and the other infinite. They are obtained from a dynamical system which is ergodic and λ-preserving measure.

Definition 4 *Let $k \in]0,1[$. For each $x \in]0,1]$, there exists a non negative integer n_0 such that*

$$k^{n_0+1} \le x \le k^{n_0}.$$

Hence, $x = k^{n_0+1} + y_1$, with $0 \le y_1 \le k^{n_0}(1-k)$; and we can write

$$x = k^{n_0+1} + k^{n_0}(1-k)\,x_1,$$

where $x_1 \in [0,1]$. Reasoning on x_1, we obtain

$$x = k^{n_0+1} + k^{n_0+n_1+1}(1-k) + k^{n_0+n_1}(1-k)^2\,x_2;$$

and, by induction, we have this formal equality:

$$x = \sum_{d=0}^{+\infty} (1-k)^d\, k^{1+\sum_{j=0}^{d} n_j},$$

which will be called the generalized dyadic representation for the corresponding number x.

These series converge to x, because

$$\left| x - \sum_{d=0}^{m} (1-k)^d\, k^{1+\sum_{j=0}^{d} n_j} \right| < (1-k)^{m+1},$$

and the majorization M-test of Weierstrass is applied. We summarize this situation:

Proposition 5 *Let $k \in]0,1[$. If $x \in]0,1]$, then there exists an increasing sequence of naturals $1 \le m_0 \le m_1 \le \cdots \le m_d \le \cdots$, such that $x = \sum_{d=0}^{+\infty} (1-k)^d\, k^{m_d}$.*

Proposition 6 *The expansion in the above proposition is unique but it would be finite or stationary (i.e., $m_d = m_j$ if $d \ge j$).*

Proof. Because

$$1 = k + k(1-k) + k(1-k)^2 + k(1-k)^3 + \cdots,$$

in the finite or stationary cases double expansions appear:

$$\sum_{d=0}^{n} (1-k)^d k^{m_d} = \sum_{d=0}^{n-1} (1-k)^d k^{m_d} + \sum_{d=n}^{+\infty} (1-k)^d k^{m_d+1}.$$

By the other hand, if the sequence (m_d) is not bounded (the expansion for x is not finite neither stationary), let us consider infinite expansions for x and y:

$$\begin{aligned} x &= \sum_{d=0}^{+\infty} (1-k)^d k^{m_d} \\ y &= \sum_{d=0}^{+\infty} (1-k)^d k^{m'_d} \end{aligned}$$

where $0 \leq d \leq n-1$, implies $m_d = m'_d$, and if $d \geq n$ then $m_d < m'_d$. Hence:

$$\begin{aligned} y &\leq \sum_{d=0}^{n-1} (1-k)^d k^{m_d} + k^{m'_n} \sum_{d=n}^{+\infty} (1-k)^d = \\ &= \sum_{d=0}^{n-1} (1-k)^d k^{m_d} + k^{m'_n - 1} (1-k)^n < \\ &< \sum_{d=0}^{n-1} (1-k)^d k^{m_d} + \sum_{d=n}^{+\infty} (1-k)^d k^{m_d} = x. \end{aligned}$$

As a consequence, for x with non bounded (m_d), x differs from y being stationary or non stationary. ■

3.2 A singular function with the generalised dyadic system

Definition 7 *For each $k \in \,]0,1[\,\backslash\{\frac{1}{2}\}$ let us define the function $f_k : [0,1] \to [0,1]$ given in the following way: each x with non stationary infinite expansion (i.e., there exist $1 \leq t_0 < t_1 < \cdots < t_d < \cdots$, such that)*

$$\begin{aligned} x \;=\; & k^{t_0} + \cdots + k^{t_0} (1-k)^{s_0} + \\ & + k^{t_1} (1-k)^{s_0+1} + \cdots + k^{t_1} (1-k)^{s_1} + \cdots \\ & + k^{t_d} (1-k)^{s_{d-1}+1} + \cdots + k^{t_d} (1-k)^{s_d} + \cdots \end{aligned}$$

is mapped to

$$\begin{aligned} & f_k(x) := k + k(1-k) + \cdots + k(1-k)^{t_0-2} + \\ & + k^{s_0+2} (1-k)^{t_0-1} + \cdots + k^{s_0+2} (1-k)^{t_1-2} + \\ & + k^{s_1+2} (1-k)^{t_1-1} + \cdots + k^{s_1+2} (1-k)^{t_2-2} + \cdots \\ & + k^{s_{d-1}+2} (1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2} (1-k)^{t_d-2} + \cdots \end{aligned}$$

If $t_0 := 1$, then $k + k(1-k) + \cdots + k(1-k)^{t_0-2}$ does not exist. In the stationary case, i.e., when x has finite expansion

$$\begin{aligned} x \;=\; & k^{t_0} + \cdots + k^{t_0} (1-k)^{s_0} + \cdots \\ & + k^{t_d} (1-k)^{s_{d-1}+1} + \cdots + k^{t_d} (1-k)^{s_d}, \end{aligned}$$

then

$$\begin{aligned} f_k(x) \;:\; & = k + k(1-k) + \cdots + k(1-k)^{t_0-2} + \cdots \\ & + k^{s_{d-1}+2} (1-k)^{t_{d-1}-1} + \cdots \\ & + k^{s_{d-1}+2} (1-k)^{t_d-2} + k^{s_d+1} (1-k)^{t_d-1}. \end{aligned}$$

Remark 8 *If we work with x given by an infinite number of chains in the form*

$$k^{t_d} (1-k)^{s_{d-1}+1} + \cdots + k^{t_d} (1-k)^{s_d},$$

the image will be an infinite sum of chains in the form

$$k^{s_{d-1}+2} (1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2} (1-k)^{t_d-2}.$$

If the number of chains is finite, the result is analogous, but adding

$$k^{s_d+1} (1-k)^{t_d-1}.$$

First, we give some properties concerning f_k.

Proposition 9 *The function f_k is a strictly decreasing function.*

Proof. Let us consider four different cases.

a. Let be $y < x$, where

$x = k^{t_0} + \cdots + k^{t_0} (1-k)^{s_0} +$

$+ k^{t_1} (1-k)^{s_0+1} + \cdots + k^{t_1} (1-k)^{s_1} + \cdots$

$+ k^{t_d} (1-k)^{s_{d-1}+1} + \cdots + k^{t_d} (1-k)^{s_d} +$

$+ k^{t_{d+1}} (1-k)^{s_d+1} + \cdots + k^{t_{d+1}} (1-k)^{s'_{d+1}} + \cdots$

$+ k^{t_{d+1}} (1-k)^{s_{d+1}} + \cdots$

and

$y = k^{t_0} + \cdots + k^{t_0} (1-k)^{s_0} +$

$+ k^{t_1} (1-k)^{s_0+1} + \cdots + k^{t_1} (1-k)^{s_1} + \cdots$

$+ k^{t_d} (1-k)^{s_{d-1}+1} + \cdots + k^{t_d} (1-k)^{s_d} +$

$+ k^{t_{d+1}} (1-k)^{s_d+1} + \cdots + k^{t_{d+1}} (1-k)^{s'_{d+1}} +$

$+ k^{t'_{d+2}} (1-k)^{s'_{d+1}+1} + \cdots$

(note that $s'_{d+1} < s_{d+1}$), thereof

$f_k(x) := k + k(1-k) + \cdots + k(1-k)^{t_0-2} + \cdots$

$+ k^{s_{d-1}+2} (1-k)^{t_{d-1}-1} + \cdots$

$+k^{s_{d-1}+2}(1-k)^{t_d-2}+k^{s_d+2}(1-k)^{t_d-1}+\cdots$

$+k^{s_d+2}(1-k)^{t_{d+1}-2}+k^{s_{d+1}+2}(1-k)^{t_{d+1}-1}+\cdots$

and

$f_k(y):=k+k(1-k)+\cdots$

$+k(1-k)^{t_0-2}+\cdots+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1}+\cdots$

$+k^{s_{d-1}+2}(1-k)^{t_d-2}+k^{s_d+2}(1-k)^{t_d-1}+\cdots$

$+k^{s_d+2}(1-k)^{t_{d+1}-2}+k^{s'_{d+1}+2}(1-k)^{t_{d+1}-1}+\cdots;$

which yields to $f_k(x)<f_k(y)$.

b. Now, let us consider x and y, $y<x$, in the form:

$x=k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+$

$+k^{t_1}(1-k)^{s_0+1}+\cdots+k^{t_1}(1-k)^{s_1}+\cdots$

$+k^{t_d}(1-k)^{s_{d-1}+1}+\cdots$

$+k^{t_d}(1-k)^{s_d}+\cdots+k^{t_{d+1}}(1-k)^{s_d+1}+\cdots$

and

$y=k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+$

$+k^{t_1}(1-k)^{s_0+1}+\cdots+k^{t_1}(1-k)^{s_1}+\cdots$

$+k^{t_d}(1-k)^{s_{d-1}+1}+\cdots+k^{t'_{d+1}}(1-k)^{s_{d+1}}+\cdots$

$+k^{t_{d+1}}(1-k)^{s'_{d+1}+1}+k^{t'_{d+1}}(1-k)^{s'_{d+1}+1}+\cdots$

(Note here that $t_{d+1}<t'_{d+1}$.) Hence,

$f_k(x):=k+k(1-k)+\cdots+k(1-k)^{t_0-2}+\cdots$

$+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1}+\cdots+k^{s_{d-1}+2}(1-k)^{t_d-2}+$

$+k^{s_d+2}(1-k)^{t_d-1}+\cdots+k^{s_d+2}(1-k)^{t_{d+1}-2}+\cdots$

and

$f_k(y):=k+k(1-k)+\cdots+k(1-k)^{t_0-2}+\cdots$

$+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1}+\cdots+k^{s_{d-1}+2}(1-k)^{t_d-2}+$

$+k^{s_d+2}(1-k)^{t_d-1}+\cdots+k^{s_d+2}(1-k)^{t'_{d+1}-2}+\cdots;$

which, again, implies $f_k(x)<f_k(y)$.

c. x has finite expansion and y has infinite expansion, with $x<y$.

d. x and y have both finite expansion.

Cases *c.* and *d.* fulfil the rest of possibilities and the proofs run as in *a.* and *b.* ■

Proposition 10 *The function f_k is a continuous function.*

Proof. Let $x\in[0,1]$, and we consider the following two cases:

a. x has not finite expansion:

$$\begin{aligned} x &= k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+\cdots \\ &\quad +k^{t_d}(1-k)^{s_{d-1}+1}+\cdots+k^{t_d}(1-k)^{s_d}+\cdots \end{aligned}$$

If a sequence (x_n) converges to x, then there exists m such that $n\geq m$ implies

$$\begin{aligned} x_n &= k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+\cdots \\ &\quad +k^{t_d}(1-k)^{s_{d-1}+1}+\cdots+k^{t_d}(1-k)^{s_d}+ \\ &\quad +k^{t'_{d+1}}(1-k)^{s_d+1}+\cdots+k^{t'_{d+1}}(1-k)^{s'_d+1}+\cdots, \end{aligned}$$

i.e., x_n and x coincide on the first d blocks if $n\geq m$. Then $f(x)$ and $f(x_n)$ have expansions that coincide in the form

$$\begin{aligned} &k+k(1-k)+\cdots+k(1-k)^{t_0-2}+\cdots \\ &+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1}+\cdots+k^{s_{d-1}+2}(1-k)^{t_d-2}+ \\ &+k^{s_d+2}(1-k)^{t_d-1}+\cdots \end{aligned}$$

which implies that

$$\begin{aligned} |f_k(x)-f_k(x_n)| &\leq 2k^{s_d+2}\left[(1-k)^{t_d-1}+(1-k)^{t_d}+\cdots\right]= \\ &= 2k^{s_d+1}(1-k)^{t_d-1}. \end{aligned}$$

But, if $n\to+\infty$ then, d,t_d and $s_d\to+\infty$. Hence, $f_k(x_n)\longrightarrow f_k(x)$.

b. If x has finite expansion, then we will consider sequences (x_n) one side converging to x. Firstly, let $x_n\searrow x$. Because

$$\begin{aligned} x &= k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+\cdots \\ &\quad +k^{t_d}(1-k)^{s_{d-1}+1}+\cdots+k^{t_d}(1-k)^{s_d}, \end{aligned}$$

then

$x_n=k^{t_0}+\cdots+k^{t_0}(1-k)^{s_0}+\cdots k^{t_d}(1-k)^{s_{d-1}+1}+\cdots$

$+k^{t_d}(1-k)^{s_d}+k^{t'_{d+1}}(1-k)^{s_d+1}+\cdots$

(where $t'_{d+j}=t'_{d+j}(n)$ for all j's and $t'_{d+j}\to+\infty$ if n does). Applying f_k on them:

$f_k(x)=k+k(1-k)+\cdots+k(1-k)^{t_0-2}+\cdots$

$+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1}+\cdots+k^{s_{d-1}+2}(1-k)^{t_d-2}+$

$+k^{s_d+1}(1-k)^{t_d-1};$

and

$f_k(x_n)=k+k(1-k)+\cdots+k(1-k)^{t_0-2}+\cdots$

$+k^{s_{d-1}+2}(1-k)^{t_{d-1}-1}+\cdots+k^{s_{d-1}+2}(1-k)^{t_d-2}+$

$+k^{s_d+2}(1-k)^{t_d-1}+\cdots+k^{s_d+2}(1-k)^{t'_{d+1}-2}+\cdots$

But, if $t'_{d+j}\to+\infty$, then $k^{s_d+2}(1-k)^{t_d-1}+\cdots+k^{s_d+2}(1-k)^{t'_{d+1}-2}+\cdots=k^{s_d+1}(1-k)^{t_d-1}$; i.e., $f_k(x_n)\longrightarrow f_k(x)$.

By the other hand, if $x_n \nearrow x$, things run analogously.

Both cases *a.* and *b.* together give the desiderable convergence $f_k(x_n) \longrightarrow f_k(x)$, if $n \to +\infty$. ■

The next lemma will be useful in that follows.

Lemma 11 *([1, Th.13]) The set of points* $x = \sum_{n=1}^{+\infty} (1-k)^n k^{m_n}$ *for which* $\lim_n \frac{m_n}{n} = \frac{k}{1-k}$ *is a set of* λ*-measure 1.*

Definition 12 *A point* x *is said normal in the basis* k *(or* k*-normal) if it verifies the property in the lemma above.*

Theorem 13 *There exists a set of measure 1 where the function* f_k *has null derivatives on each point.*

Proof. Let us consider x a normal number in this representation system. It will necessarily have an infinite expansion:

$$\begin{aligned} x &= k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots \\ &\quad + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots; \end{aligned}$$

and we define sequences (x_n) and (y_n):

$$\begin{aligned} x_n &= k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots \\ &\quad + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d}, \end{aligned}$$

and

$$y_n = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots$$

$$+ k^{t_d}(1-k)^{s_d} + k^{t_d}(1-k)^{s_d+1},$$

verifying $x_n \le x \le y_n$ for all $n \in \mathbb{N}$. The function f_k acts respectively as follows on them:

$$f_k(x_n) = k + k(1-k) + \cdots + k(1-k)^{t_0-2} + \cdots$$

$$+ k^{s_{d-1}+2}(1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2}(1-k)^{t_d-2} +$$

$$+ k^{s_d+1}(1-k)^{t_d-1};$$

and

$$f_k(y_n) = k + k(1-k) + \cdots + k(1-k)^{t_0-2} + \cdots$$

$$+ k^{s_{d-1}+2}(1-k)^{t_{d-1}-1} + \cdots + k^{s_{d-1}+2}(1-k)^{t_d-2} +$$

$$+ k^{s_d+2}(1-k)^{t_d-1}.$$

Then, we have

$$\begin{aligned} &\tfrac{f_k(y_n)-f_k(x_n)}{y_n-x_n} = \\ &= \tfrac{k^{s_d+2}(1-k)^{t_d-1} - k^{s_d+2}(1-k)^{t_d-1}}{k^{t_d}(1-k)^{s_d+1}} \\ &= \tfrac{-k^{s_d+1}(1-k)^{t_d}}{k^{t_d}(1-k)^{s_d+1}} = -\left(\tfrac{k}{1-k}\right)^{\frac{1-2k}{1-k}s_d + o(s_d)}, \end{aligned}$$

where we have taken into account that $t_d \simeq \frac{k}{1-k} s_d$ for normal numbers.

Now, on the one hand, if $k > 1/2$, then $\frac{1-2k}{1-k} < 0$, and

$$\lim \frac{f_k(y_n) - f_k(x_n)}{y_n - x_n} = 0.$$

On the other hand, if $k < 1/2$, then $\frac{1-2k}{1-k} > 0$, and, analogously,

$$\lim \frac{f_k(y_n) - f_k(x_n)}{y_n - x_n} = 0.$$

Hence, the existence of $f_k'(x)$ implies it is null.

But monotonicity of f means its derivability on a set of measure 1 (see [11]). Hence, $f_k'(x) = 0$ on a set of measure 1. ■

Proposition 14 *The function* f_k *does not admit non-zero derivatives.*

Proof. Let us consider x with finite expansion:

$$x = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d},$$

and

$$x_n = k^{t_0} + \cdots + k^{t_d}(1-k)^{s_d} + k^n(1-k)^{s_d+1}.$$

For each n:

$$\begin{aligned} \frac{f_k(x_n) - f_k(x)}{x_n - x} &= \frac{-k^{s_d+1}(1-k)^m}{k^m(1-k)^{s_d+1}} \\ &= -\left(\frac{k}{1-k}\right)^{s_d+1-m}, \end{aligned}$$

and when $n \to +\infty$, this sequence goes to 0 or to $-\infty$, depending on the value of $\frac{k}{1-k}$. Hence, in case the limit existing, it must be zero.

On the other hand, if x has not finite expansion, then let us consider sequences (x_n) and (y_n), as in the above theorem. In addition, let

$$\begin{aligned} y_n' &= k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + \cdots \\ &\quad + k^{t_d}(1-k)^{s_{d-1}+1} + \cdots + k^{t_d}(1-k)^{s_d} + \\ &\quad + k^{t_d}(1-k)^{s_d+1} + k^{t_d}(1-k)^{s_d+2}. \end{aligned}$$

For these numbers:

$$\frac{f_k(y_d) - f_k(x_d)}{y_d - x_d} = -\left(\frac{k}{1-k}\right)^{-s_d+1-t_d},$$

and

$$\frac{f_k(y_d') - f_k(x_d)}{y_d' - x_d} = -\frac{1+k}{2-k}\left(\frac{k}{1-k}\right)^{-s_d+1-t_d}.$$

In case there exists derivative, the quotient of both limits $-\frac{1+k}{2-k}$ must be equal to 1; but this is not the case when $k \neq 1/2$. Hence, because the derivative exists, it must be equal to 0. ■

Theorem 15 *The functon f_k is the unique solution for the system of functional equations given by*

$$\begin{cases} f(kx) = k + (1-k) f(x) \\ f(k + (1-k)x) = kf(x) \end{cases}$$

Proof. Firstly, we will show that f_k verifies this system of functional equations. Let us consider the case when x has infinite expansion; the finite case runs analogously. If

$$x = k^{t_0} + \cdots + k^{t_0}(1-k)^{s_0} + k^{t_1}(1-k)^{s_0+1} + \cdots + k^{t_d}(1-k)^{s_d} + \cdots,$$

then

$$f_k(x) := k + k(1-k) + \cdots + k(1-k)^{t_0-2} +$$

$$+ k^{s_0+2}(1-k)^{t_0-1} + \cdots + k^{s_0+2}(1-k)^{t_1-2} +$$

$$+ k^{s_1+2}(1-k)^{t_1-1} + \cdots + k^{s_1+2}(1-k)^{t_2-2} + \cdots$$

For these formulae, writing kx and $f_k(kx)$, the first of the two equation holds.

For the second equation the reasoning is that follows: if $t_0 \neq 1$, then

$$k + (1-k)x = k + k^{t_0}(1-k) + \cdots + k^{t_0}(1-k)^{s_0+1} +$$

$$+ k^{t_1}(1-k)^{s_0+2} + \cdots + k^{t_1}(1-k)^{s_1+1} + \cdots$$

and

$$f_k(k + (1-k)x) = k^2 + k^2(1-k) + \cdots$$

$$+ k^2(1-k)^{t_0-2} + k^{s_0+3}(1-k)^{t_0-1} + \cdots$$

$$+ k^{s_0+3}(1-k)^{t_1-2} + \cdots$$

and if $t_0 = 1$, then

$$f_k(x) = k^{s_0+2}(1-k)^{t_0-1} + \cdots + k^{s_0+2}(1-k)^{t_1-2} + \cdots$$

and

$$f_k(k + (1-k)x) = k^{s_0+3}(1-k)^{t_0-1} + \cdots + k^{s_0+3}(1-k)^{t_1-2} + \cdots$$

For both cases, the second of the equations is fulfilled. ■

Corollary 16 *The functions n_k and f_k are, in fact, the same function.*

In [2], the authors show two important properties concerning k-negations. Here λ denotes de Lebesgue measure on the reals.

Corollary 17 *The k-negation n_k maps a set of λ-measure 0 on a set of λ-measure 1. The initial set has Hausdorff dimension equal to* $\frac{\ln\left[k^k(1-k)^{1-k}\right]}{\ln\left[k^{1-k}(1-k)^k\right]}$.

Corollary 18 *The k-negation n_k maps a set of λ-measure 1 on a set of λ-measure 0 with fractal dimension equal to* $\frac{\ln\left[k^k(1-k)^{1-k}\right]}{\ln\left[k^{1-k}(1-k)^k\right]}$.

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