A complete logic for fuzzy functional dependencies over t-norms

P. Cordero[1] **M. Enciso**[1] **A. Mora**[1] **I. P. de Guzmán**[1]

[1] Universidad de Málaga, Spain, {pcordero,enciso}@uma.es, {amora,guzman}@ctima.uma.es

Abstract

A fuzzification of the concept of functional dependency based on t-norms is presented. T-norms are equally used in the definition of the similarity relations in the domains and in the fuzziness degree assigned to functional dependencies. A sound and complete logic for this fuzzy functional dependencies is also introduced and, finally, its applicability to redundancy removing is shown.

Keywords: Fuzzy Logic, Fuzzy Functional Dependency, T-norms.

1 Introduction

The developments of Fuzzy Logic during the last three decades have allowed the generalization of the relational model in order to consider imprecise knowledge. L. Zadeh affirms in [25] that the relational facet -FLr- of the Fuzzy Logic is focussed on fuzzy relations and fuzzy dependencies. In FLr facet, L. Zadeh refers to some works devoted to the notion of Fuzzy Attribute Implication [5]. This notion corresponds with the idea of Fuzzy Functional Dependency (FFD)

Three approaches have been used in the literature [22] to consider fuzzy databases: the use of fuzzy membership values [1, 14], the introduction of possibility distributions [18–20] or similarity relations [7–9].

The last approach has been fully tackled. Functional Dependencies have been studied over Domains with Similarity Relations and some definitions of Fuzzy Functional Dependency (FFD) have been introduced [3, 22–24]. In these works, some complete axiomatic systems defined over FFDs with similarity relations are defined too. They are extensions of classical FD logics and lack of an executable perspective. They were designed to be a formal framework to show the semantics of the functional dependency (either classical or fuzzy). As a matter of fact, they have not been used to reason about functional dependencies and it is not a trivial matter to design an automatic method directly based on their axiomatic system. The main obstacle is that all of them are strongly based on the transitivity axiom, which limits the efficiency of the automatic deduction methods for these logics.

In [13] we have introduced for the first time a sound and complete axiomatic system for Fuzzy Functional Dependencies where the core is a Fuzzy Simplification Rule instead of the usual Fuzzy Transitivity Rule. The importance of our logic comes from the development of executable algorithms to reason automatically and solve the implication problem with FFDs [12], and how the simplication rule for FFDs may be used to remove redundancy from a set of FFDs [13].

A similar approach is tackled by for Belohlavek et.al. in [2,4]. They show a system of Armstrong-like derivation rules and describe a non-redundant basis of all rules which are true in a data table and present a method for computation of nonredundant bases of attribute implications from data tables with fuzzy attributes. Dependency of attributes and determining redundant attributes in decision tables are some of the important issues in the application of Knowledge Discovery and Data Mining. In [15] the authors introduce the fuzzy functional dependency that satisfies Armstrong Axioms. In addition, they also discuss some interesting applications such as approximate data reduction.

Some applications require the use of new fuzzy operators. They are the triangular-norms (t-norms) used in the context of information retrieval with fuzzy ontologies. The t-norm concept is central in fuzzy logic and its study is very important from the theoretical point of view to the applications [11].

In this work, we present a fuzzification of the concept of functional dependency based on t-norms and we introduce a sound and complete logic for this kind of fuzzy functional dependencies. Our approach (Section 3) is given by considering similarity relations in the domains. These similarity relations are fuzzy equivalence relations in its broadest sense (based on t-norms). Moreover, the concept of FFD introduced is actually fuzzy, that is, they assigns truth values in the $[0,1]$ interval and generalizes other approaches given in the literature [13, 22] by replacing the minimum operator by arbitrary t-norms. In Section 4, we introduce our logic and prove its soundness and completeness. Finally, in Section 4.3, we show that our logic is executable and its principal inference rule can be seen as a simplification tool because they directly remove redundancy.

2 Preliminaries

We present here the basic concepts about functional dependencies as they appear in database literature. Let $\{D_a \mid a \in \mathcal{A}\}$ be a family of sets indexed in a finite non-empty set of indexes $\mathcal{A}$. We call *attributes* to the indexes and *domain of the attribute* a to the set D_a. We work over the product of these domains, $\mathbb{D} = \prod_{a\in\mathcal{A}} D_a$. The elements in this product $t = (t_a)_{a\in\mathcal{A}} \in \mathbb{D}$ will be named tuples. A relation is a set of tuples $R \subseteq \mathbb{D}$, usually represented as a table.

We introduce here the notation widely accepted in the database community. Given $X, Y \subseteq \mathcal{A}$, XY denotes $X \cup Y$. Given $X \subseteq \mathcal{A}$, D_X denotes $\prod_{a\in X} D_a$. Let $t \in R$ be a tuple, then $t_{/X}$ denotes the *projection* of t to D_X; that is, if $t = (t_a)_{a\in\mathcal{A}}$ then $t_{/X} = (t_a)_{a\in X}$.

Definition 2.1 *Any statement of the type $X \to Y$, where $X, Y \subseteq \mathcal{A}$, is named a* **functional dependency**. *We say that a relation $R \in \mathbb{D}$ satisfies $X \to Y$ if, for all $t_1, t_2 \in R$ we have that: $t_{1/X} = t_{2/X}$ implies that $t_{1/Y} = t_{2/Y}$.*

Remark 2.1 *The term* functional *comes from the fact that: the relation R satisfies the FD $X \to Y$ if R restricted to XY is a (partial) function from D_X to D_Y.*

The aim of this paper is to fuzzify this concept and to give a sound and complete fuzzy logic that allows us to manipulate this kind of functional dependencies. Our approach uses the unit interval $[0,1]$ for the system of truth values, the infimum (denoted by $\wedge$) as the universal quantifier, the supremun (denoted by $\vee$) as the existential quantifier and an arbitrary t-norm (triangular norm denoted by $\otimes$) as the conjunction. That is, the system of truth values is $([0,1], \vee, \wedge, 0, 1, \otimes)$ where $([0,1], \otimes, 1)$ is a commutative monoid and $\otimes$ is monotonic ($a \leq b$ implies $a \otimes c \leq b \otimes c$, for all $a, b, c \in [0,1]$)

3 Fuzzy Functional Dependencies

The most usual way to fuzzify the concept of functional dependency is by replacing the equality in the definition by fuzzy relations named similarity relations. In this case, we will consider that each domain D_a is endowed with a similarity relation $\rho_a : D_a \times D_a \to [0,1]$, that is, a reflexive, symmetric and $\otimes$-transitive fuzzy relation. We can extend these relations to D_X for all $X \subseteq \mathcal{A}$ as follow:

$$\rho_X(t_1, t_2) = \bigwedge_{a\in X} \rho_a(t_1, t_2)$$ [1]

Remark 3.1 *The definitions of fuzzy functional dependency in the literature [3, 12, 21, 23, 24] are very similar, having slight differences among them. They fuzzify the equality between the attributes value in the following way:* A relation $R \subseteq \mathbb{D}$ satisfies the FFD $X \to Y$ if $\rho_Y(t_1, t_2) \geq \rho_X(t_1, t_2)$ holds, for all $t_1, t_2 \in R$.

Although the definition introduces the fuzzy relations of similarity and generalizes the classical definition, we can say that the functional dependency remains crisp. The inclusion of a degree of fuzzyness in the dependency itself is done in [22]. In this work we adopt the following definition of fuzzy functional dependency.

Definition 3.1 *A* ***fuzzy functional dependency*** *is an expression $X \xrightarrow{\theta} Y$ where $X, Y \subseteq \mathcal{A}$ and $\theta \in [0,1]$. A relation $R \subseteq \mathbb{D}$ is said to satisfy $X \xrightarrow{\theta} Y$ if, for all $t_1, t_2 \in R$, the inequation $\theta \otimes \rho_X(t_1, t_2) \leq \rho_Y(t_1, t_2)$ holds.*

We remark that, if $\theta = 1$, the previous definition of FFD matches up with the definition of FFD proposed in Remark 3.1. Moreover, if the similarity relations are strongly reflexive then it define a crisp classical functional dependency.

Proposition 3.2 *Let $R \subseteq \mathbb{D}$, $X, Y, Y' \subseteq \mathcal{A}$ and $\theta_1, \theta_2 \in [0,1]$.*

1. *If $Y \subseteq X$ then R satisfies $X \xrightarrow{1} Y$.*
2. *If R satisfies $X \xrightarrow{\theta_1} Y$ and $\theta_1 \geq \theta_2$ then R satisfies $X \xrightarrow{\theta_2} Y$.*
3. *If R satisfies $X \xrightarrow{\theta_1} Y$ and $Y' \subseteq Y$ then R satisfies $X \xrightarrow{\theta_1} Y'$.*

[1] To simplify the notation, when no confusion arise, we write $\rho_X(t_1, t_2)$ instead of $\rho_X(t_{1/X}, t_{2/X})$.

PROOF: Straightforward from definition. □

Proposition 3.3 (Composition) *Let* $R \subseteq \mathbb{D}$, $X, Y, U, V \subseteq \mathcal{A}$ *and* $\theta_1, \theta_2 \in [0,1]$. *If* R *satisfies* $X \xrightarrow{\theta_1} Y$ *and* $U \xrightarrow{\theta_2} V$ *then* R *also satisfies* $XU \xrightarrow{\theta_1 \wedge \theta_2} YV$.

PROOF: If R satisfies $X \xrightarrow{\theta_1} Y$ and $U \xrightarrow{\theta_2} V$ then, for all $t_1, t_2 \in R$, $\rho_Y(t_1,t_2) \geq \theta_1 \otimes \rho_X(t_1,t_2)$ and $\rho_U(t_1,t_2) \geq \theta_2 \otimes \rho_V(t_1,t_2)$. Therefore

$$\begin{aligned}
\rho_{YV}(t_1,t_2) &= \rho_Y(t_1,t_2) \wedge \rho_V(t_1,t_2) \\
&\geq \big(\theta_1 \otimes \rho_X(t_1,t_2)\big) \wedge \big(\theta_2 \otimes \rho_U(t_1,t_2)\big) \\
&\overset{(1)}{\geq} (\theta_1 \wedge \theta_2) \otimes (\rho_X(t_1,t_2) \wedge \rho_U(t_1,t_2)) \\
&= (\theta_1 \wedge \theta_2) \otimes \rho_{XU}(t_1,t_2)
\end{aligned}$$

where in (1) we have used that, for isotonie of $\otimes$,

$$\theta_1 \otimes \rho_X(t_1,t_2) \geq (\theta_1 \wedge \theta_2) \otimes (\rho_X(t_1,t_2) \wedge \rho_U(t_1,t_2))$$
$$\theta_2 \otimes \rho_U(t_1,t_2) \geq (\theta_1 \wedge \theta_2) \otimes (\rho_X(t_1,t_2) \wedge \rho_U(t_1,t_2))$$

□

Proposition 3.4 (Simplification) *Let* $R \subseteq \mathbb{D}$, $X, Y, U, V \subseteq \mathcal{A}$ *and* $\theta_1, \theta_2 \in [0,1]$. *If* $X \subseteq U$, $X \cap Y = \varnothing$ *and* R *satisfies* $X \xrightarrow{\theta_1} Y$ *and* $U \xrightarrow{\theta_2} V$ *then* R *also satisfies* $U\text{-}Y \xrightarrow{\theta_1 \otimes \theta_2} V\text{-}Y$.

PROOF: The set U can be written as disjoin union as $U = X \cup U_Y \cup U_r$ where $U_Y = U \cap Y$ and $U_r = U \smallsetminus (X \cup Y)$.

From hypothesis, for all $t_1, t_2 \in R$,

$$\rho_Y(t_1,t_2) \geq \theta_1 \otimes \rho_X(t_1,t_2) \quad (1)$$

$$\rho_V(t_1,t_2) \geq \theta_2 \otimes \rho_U(t_1,t_2) \quad (2)$$

and, by replacing (1) in (2), we have that

$$\rho_{V\text{-}Y}(t_1,t_2) \overset{(i)}{\geq} \rho_V(t_1,t_2) \overset{(2)}{\geq} \theta_2 \otimes \rho_U(t_1,t_2)$$
$$= \theta_2 \otimes (\rho_X(t_1,t_2) \wedge \rho_{U_Y}(t_1,t_2) \wedge \rho_{U_r}(t_1,t_2))$$
$$\overset{(ii)}{\geq} \theta_2 \otimes (\rho_X(t_1,t_2) \wedge \rho_Y(t_1,t_2) \wedge \rho_{U_r}(t_1,t_2))$$
$$\overset{(1)}{\geq} \theta_2 \otimes (\rho_X(t_1,t_2) \wedge (\theta_1 \otimes \rho_X(t_1,t_2)) \wedge \rho_{U_r}(t_1,t_2))$$

where (i) is due to $V \smallsetminus Y \subseteq V$ and (ii) is due to $U_Y = U \cap Y \subseteq Y$ and the isotonie of $\otimes$ and $\wedge$. Now, since $\rho_X(t_1,t_2) \geq \theta_1 \otimes \rho_X(t_1,t_2)$,

$$\rho_{V\text{-}Y}(t_1,t_2) \geq \theta_2 \otimes ((\theta_1 \otimes \rho_X(t_1,t_2)) \wedge \rho_{U_r}(t_1,t_2)) \quad (3)$$

Since $\otimes$ is monotonic, $\theta_1 \otimes \rho_X(t_1,t_2) \geq \theta_1 \otimes (\rho_X(t_1,t_2) \wedge \rho_{U_r}(t_1,t_2))$ and $\rho_{U_r}(t_1,t_2) = 1 \otimes \rho_{U_r}(t_1,t_2) \geq \theta_1 \otimes (\rho_X(t_1,t_2) \wedge \rho_{U_r}(t_1,t_2))$. Therefore,

$$\begin{aligned}
(\theta_1 \otimes \rho_X(t_1,t_2)) \wedge \rho_{U_r}(t_1,t_2) & \\
&\geq \theta_1 \otimes (\rho_X(t_1,t_2) \wedge \rho_{U_r}(t_1,t_2)) \\
&= \theta_1 \otimes \rho_{U\text{-}Y}(t_1,t_2)
\end{aligned}$$

Finally, by replacing in (3), and using commutativity and associativity of $\otimes$,

$$\rho_{V\text{-}Y}(t_1,t_2) \geq (\theta_1 \otimes \theta_2) \otimes \rho_{U\text{-}Y}(t_1,t_2)$$

□

4 The Simplification Logic for FFDs

We are interested in an axiomatic system that allows us to syntactically derive FFDs. There exists in the literature some complete axiomatic system defined over FFDs with similarity relations [3, 23, 24]. However, there are not many axiomatic systems in the literature to reasoning with fuzzy functional dependencies where the dependency is fuzzy. One of them is given by Sozat and Yazici [22]. It is a fuzzy extension of Armstrong Axiom's and it has the problem inherent in transitivity.

4.1 $\mathbf{SL}_{FFD}$ Logic

We introduce $\mathbf{SL}_{FFD}$, a new logic more adequate for the applications, named Simplification Logic for fuzzy functional dependencies. Its language is the following:

Definition 4.1 *Given a numerable set of attribute symbols* $\mathcal{A}$, *we define the language* $\mathbf{L} = \{X \xrightarrow{\theta} Y \mid \theta \in [0,1]$ *and* $X, Y \in 2^{\mathcal{A}}$ *with* $X \neq \varnothing\}$.

The semantic of this logic was outlined in previous sections. The semantic models are pairs made up of a family of domains with its similarities $\{(D_a, \rho_a) \mid a \in \mathcal{A}\}$ and a relation $R \subseteq \mathbb{D}$. However, to simplify the notation, we will only refer to the relation. So $R \models X \xrightarrow{\theta} Y$ denotes that R satisfies the functional dependency $X \xrightarrow{\theta} Y$, $R \models \Gamma$ denotes that R satisfies every fuzzy functional dependency in the set Γ and $\Gamma \models X \xrightarrow{\theta} Y$ denotes that, for all $R \subseteq \mathbb{D}$, $R \models \Gamma$ implies $R \models X \xrightarrow{\theta} Y$. In this point we present the axiomatic system:

Definition 4.2 *The axiomatic system* $\mathcal{S}_{FFD}$ *on* $\mathbf{L}$ *has one axiom scheme:* [2]

[2] In the literature, the set of attributes Y must be nonempty. In $\mathbf{SL}_{FD}$, we consider the empty attribute, denoted $\top$. Note that $X \xrightarrow{1} \top$ is an axiom scheme.

*Reflexive Axioms (***Ax***): for all* $Y \subseteq X$

$$\vdash_{\mathcal{S}_{FFD}} X \xrightarrow{1} Y$$

The inferences rules are the following:

*Inclusion Rule (***InR***): if* $\theta_1 \geq \theta_2$

$$X \xrightarrow{\theta_1} Y \ \vdash_{\mathcal{S}_{FFD}} X \xrightarrow{\theta_2} Y$$

*Decomposition Rule (***DeR***): if* $Y' \subseteq Y$

$$X \xrightarrow{\theta} Y \vdash_{\mathcal{S}_{FFD}} X \xrightarrow{\theta} Y'$$

*Composition Rule (***CoR***):*

$$X \xrightarrow{\theta_1} Y,\ U \xrightarrow{\theta_2} V \ \vdash_{\mathcal{S}_{FFD}} XU \xrightarrow{\theta_1 \wedge \theta_2} YV$$

*Simplification Rule (***SiR***): if* $X \subseteq U$ *and* $X \cap Y = \varnothing$

$$X \xrightarrow{\theta_1} Y,\ U \xrightarrow{\theta_2} V \ \vdash_{\mathcal{S}_{FFD}} U\text{-}Y \xrightarrow{\theta_1 \otimes \theta_2} V\text{-}Y$$

The *deduction* ($\vdash_{\mathcal{S}_{FFD}}$) and *equivalence* ($\equiv_{\mathcal{S}_{FFD}}$) concepts are introduced as usual.

4.2 Soundness and Completeness of $\mathbf{SL}_{FFD}$ Logic

To study the implication between the syntactic and the semantic level requires several previous definitions.

Definition 4.3 *Let* Γ *be a set of fuzzy functional dependencies over* $\mathcal{A}$*. The* closure *of* Γ *is the set* $\Gamma^+ = \{X \xrightarrow{\theta} Y \mid \Gamma \vdash_{\mathcal{S}_{FFD}} X \xrightarrow{\theta} Y\}$.

Note that, as a consequence of **Ax** and **InR**, Γ^+ assigns an infinite set of pairs (Y, θ) to every non-empty set X. If the set Y is also fixed then Γ^+ gives an interval (consequence of **InR**) whose supremum will be denoted as $\theta^+_{X,Y}$

$$\theta^+_{X,Y} = \sup\{\theta \in [0,1] \mid X \xrightarrow{\theta} Y \in \Gamma^+\}$$

On the other hand, if we fix the value of θ then a subset of $2^{\mathcal{A}}$ is obtained. This set is finite and, by **DeR** and **CoR**, is an ideal of $(2^{\mathcal{A}}, \subseteq)$. The maximum element of this ideal will be denoted by X^+_θ.

$$X^+_\theta = \max\{Y \subseteq \mathcal{A} \mid X \xrightarrow{\theta} Y \in \Gamma^+\}$$

The following theorem ensures the soundness and completeness of the axiomatic system.

Theorem 4.4 *Let* Γ *be a finite set of fuzzy functional dependencies over* $\mathcal{A}$, $\varnothing \neq X, Y \subseteq \mathcal{A}$ *and* $\theta \in [0,1]$*. Then* $\Gamma \vdash_{\mathcal{S}_{FFD}} X \xrightarrow{\theta} Y$ *if and only if* $\Gamma \models X \xrightarrow{\theta} Y$.

PROOF: The soundness is a consequence of Propositions 3.2, 3.3 and 3.4. The completeness is proved showing that $\Gamma \not\vdash_{\mathcal{S}_{FFD}} X \xrightarrow{\theta} Y$ implies that $\Gamma \not\models X \xrightarrow{\theta} Y$. That is, if $Y \not\subseteq X^+_\theta$ then there exists a model for Γ that it is not model for $X \xrightarrow{\theta} Y$.

Let $\Theta = \{\theta^+_{U,V} \mid U, V \in 2^{\mathcal{A}}, U \neq \varnothing \text{ and } \theta^+_{U,V} < \theta\}$ and let $\tau \in [0,1]$ such that $\theta > \tau > \max\Theta$. Let $D_a = \{u, v\}$, $\rho_a(u,u) = \rho_a(v,v) = 1$ and $\rho_a(u,v) = \rho_a(v,u) = \tau$ for all $a \in \mathcal{A}$. We will prove that the relation $R = \{t_1, t_2\} \subseteq \prod_{a \in \mathcal{A}} D_a$ where $t_{1a} = u$, for all $a \in \mathcal{A}$, $t_{2a} = u$ for all $a \in X^+_\theta$, and $t_{2a} = v$, for all $a \notin X^+_\theta$, is a model for Γ but it is not model for $X \xrightarrow{\theta} Y$.

Let $U \xrightarrow{\theta_1} V \in \Gamma$. If $\theta^+_{U,V} < \theta$ then

$$\theta_1 \otimes \rho_U(t_1, t_2) \leq \theta_1 \leq \theta^+_{U,V} < \tau = \rho_V(t_1, t_2)$$

If $\theta^+_{U,V} \geq \theta$ and $U \not\subseteq X^+_\theta$ then $\rho_U(t_1, t_2) = \tau$ and

$$\theta_1 \otimes \rho_U(t_1, t_2) \leq \tau = \rho_V(t_1, t_2)$$

If $\theta^+_{U,V} \geq \theta$ and $U \subseteq X^+_\theta$ then $V \subseteq X^+_\theta$ (it is a consequence of the composition rule) and

$$\theta_1 \otimes \rho_U(t_1, t_2) = \theta_1 \otimes 1 = \theta_1 \leq \rho_V(t_1, t_2) = 1$$

Therefore, R is a model for Γ but it is not a model for $X \xrightarrow{\theta} Y$ because

$$\theta \otimes \rho_X(t_1, t_2) = \theta \otimes 1 = \theta > \rho_Y(t_1, t_2) = \tau$$

□

4.3 Redundancy elimination via Simplification

In a database system we look for designs with no redundancy and functional dependencies were defined to capture some semantics of the data strongly connected with the occurrence of redundancy in a database. In the same sense, redundancy is not desirable in the integrity constrain of a database.

The logic that we have introduced is more adequate for the applications. In the following, we illustrate this assertion by showing its good behavior for removing redundancy. The primitive rules allow us to directly eliminate redundancy without use other tools with upper cost. It is possible because our inference rules are really equivalences if redundancy exists, as the following theorem shows. The systematically application of the following equivalences removes redundancy.

Theorem 4.5 *Let* $X \xrightarrow{\theta_1} Y, U \xrightarrow{\theta_2} V \in \mathbf{L}$.

Decomposition equivalence: If $X \cap Y \neq \varnothing$*, then* $\{X \xrightarrow{\theta_1} Y\} \equiv_{\mathcal{S}_{FFD}} \{X \xrightarrow{\theta_1} Y\text{-}X\}$.

Simplification equivalence: If $X \cap Y = \varnothing$, $X \subseteq U$ *and* $\theta_1 \geq \theta_2$, *then*

$$\{X \xrightarrow{\theta_1} Y, U \xrightarrow{\theta_2} V\} \equiv_{\mathcal{S}_{FFD}} \{X \xrightarrow{\theta_1} Y, U\text{-}Y \xrightarrow{\theta_2} V\text{-}Y\}$$

PROOF: The decomposition equivalence is straightforward. In the second equivalence, the left-right implication is due to **SiR** and the proof of the converse implication is the following:

1. $U \xrightarrow{1} \top$	**Ax**
2. $X \xrightarrow{\theta_1} Y$	Hypothesis
3. $U \xrightarrow{\theta_1} Y$	1, 2, **CoR**
4. $U\text{-}Y \xrightarrow{\theta_2} V\text{-}Y$	Hypothesis
5. $U \xrightarrow{\theta_2} VY$	3, 4, **CoR**
6. $U \xrightarrow{\theta_2} V$	5, **DeR**

□

5 Conclusions and future works

We have presented the most generalized version of fuzzy functional dependency, considering domain of data with similarity relations in its broadest sense (based on t-norms) and including a threshold for the fuzziness of the dependency itself. We have also introduced logic for this kind of fuzzy functional dependencies proving its soundness and completeness and we show that our logic is executable. Its principal inference rule can be seen as a simplification tool because they directly remove redundancy.

As future works in the use of $\mathbf{SL}_{FFD}$ logic, we are developing an algorithm to automatically remove the redundancy of FFDs. We will also study an extension of the closure algorithm and we will face on with the implication problem using the rules of the $\mathbf{SL}_{FFD}$ logic.

Acknowledgements

This work has been partially supported by projects TIN2007-65819 (Science Ministry of Spain) and P06-FQM-02049 (Junta de Andalucía).

References

[1] J.F. Baldwin. Knowledge engineering using a fuzzy relational inference language. *Proc. IFAC Conf. on Fuzzy Information, Knowledge Representation, and Decision Processes, Marseille, France*, 15–23, 1983.

[2] R. Belohlavek, V. Vychodil. Computing non-redundant bases of if-then rules from data tables with graded attributes. *Australian Conference on Artificial Intelligence* 1126–1129 (2005)

[3] R. Belohlavek, V. Vychodil. Tables with Similarity Relations: Functional Dependencies, Complete Rules and Non-redundant Bases in Database Systems for Advanced Applications. *LNCS*, 3882: 644–658, 2006.

[4] R. Belohlavek, V. Vychodil. Data Tables with Similarity Relations: Functional Dependencies, Complete Rules and Non-redundant Bases. *DASFAA* 644–658 (2006)

[5] R. Belohlavek, V. Vychodil. Basic Algorithm for Attribute Implications and Functional Dependencies in Graded Setting. *Int. J. Found. Comput. Sci.* 19(2): 297-317 (2008)

[6] F.Bobillo, U.Straccia. Fuzzy description logics with general t-norms and datatypes. *Fuzzy Sets and Systems*, doi:10.1016/j.fss.2009.03.006.

[7] B.P. Buckles, F.E. Petry. A fuzzy representation of data for relational databases. *Fuzzy Sets and Systems*, 7: 213–216, 1982.

[8] B.P. Buckles, F.E. Petry. Fuzzy databases and their applications. *Fuzzy Inform. Decision Process.*: 361–371, 1982.

[9] B.P. Buckles, F.E. Petry. Uncertainty models in information and database systems. *J. Inform. Sci.*, 11: 77-87, 1985.

[10] Pablo Cordero, Manuel Enciso, Inmaculada P. de Guzmán, and Angel Mora. Slfd logic: Elimination of data redundancy in knowledge representation. *LNAI 2527, Springer-Verlag*, 141–150, 2002.

[11] Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna, Carles Noguera Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies *Annals of Pure and Applied Logic*, 160 (1), 53–81, 2009.

[12] Pablo Cordero, Angel Mora, Manuel Enciso, Gabriel Aguilera, I.P. de Guzmán. Reasoning about Fuzzy Functional Dependencies. *ESTYLF*: 121–126, 2008.

[13] Pablo Cordero, Manuel Enciso, Angel Mora, Gabriel Aguilera, I.P. de Guzmán. A Complete Logic for Fuzzy Functional Dependencies over Domains with Similarity Relations. *Lecture Notes In Computer Science*: 5517, 261–269, 2009.

[14] C. Giardina, I. Sack, D. Sinha. Fuzzy Field Relational Database. *Report 8332, Elect. Engng. and Computer Science Dept., Stevens Institute of Technology, Hoboken, NJ,*, 1983.

[15] Rolly Intan, Masao Mukaidono. Fuzzy Conditional Probability Relations and their Applications in Fuzzy Information Systems *Knowledge and Information Systems*: 6, 345–365, 2004.

[16] Angel Mora, Manuel Enciso, Pablo Cordero, and Inmaculada P. de Guzmán. The functional dependence implication problem: optimality and minimality. an efficient preprocessing transformation based on the substitution paradigm. *LNAI, Springer-Verlag*, 3040, 2004.

[17] Angel Mora, Gabriel Aguilera, Manuel Enciso, Pablo Cordero, and Inmaculada P. de Guzmán. A new closure algorithm based in logic: SLFD-Closure versus classical closures. *Inteligencia Artificial, Revista Iberoamericana de IA*, 31 (10): 31–40, 2006

[18] H. Prade. The connection between Lipskios approach to incomplete information data bases and Zadeh's Possibility Theory. *Proc. Int. Conf. Systems Meth.*: 402–408, 1982.

[19] H. Prade, C. Testemale. Generalizing database realtional algebra for the treatment of incomplete-uncertain information and vague queries. *2nd NAFIP Workshop*, 1983.

[20] H. Prade, C. Testemale. Generalizing database realtional algebra for the treatment of incomplete or uncertain information and vague queries. *Information Sciences*, 34: 115–143, 1984.

[21] H.V.S.V.N. Raju, A.K. Mazumdar. Fuzzy dependencies and losless join descomposition of fuzzy relational database systems. *ACM Transaction of Database Systems*, 13 (2): 129–166, 1988.

[22] Mustafa Ilker Sozat and Adnan Yazici. A complete axiomatization for fuzzy functional and multivalued dependencies in fuzzy database relations. *Fuzzy Sets and Systems*, 117 (2):161–181, 2001.

[23] B.K. Tyagi, A. Sharfuddin, R.N. Dutta, Devendra K. Tayal. A complete axiomatization of fuzzy functional dependencies using fuzzy function. *Fuzzy Set and Systems*, 151(2): 363–379, 2005.

[24] Sadok Ben Yahia, Habib Ounalli, Ali Jaoua. An Extension of Classical Functional Dependency: Dynamic Fuzzy Functional Dependency. *Information Science*, 119 (3-4):219–234, 1999.

[25] Zadeh, Lotfi, Is there a need for fuzzy logic? *Information Science*, 178(13): 2751–2779, 2008.