USING HEAVY AGGREGATIONS IN A UNIFIED MODEL BETWEEN THE WEIGHTED AVERAGE AND THE OWA OPERATOR

José M. Merigó **Montserrat Casanovas**

Department of Business Administration, University of Barcelona,
Av. Diagonal 690, 08034 Barcelona, Spain, {jmerigo, mcasanovas}@ub.edu

Abstract

We introduce a new aggregation operator called the heavy ordered weighted averaging – weighted averaging (HOWAWA) operator. It is a new aggregation operator that uses the weighted average and the ordered weighted average in the same formulation and considering the degree of importance that each concept has in the analysis. Moreover, by using heavy aggregations, we are allowing the weighting vectors to range from the minimum to the total operator. Thus, we can consider a lot of different particular cases such as the heavy weighted average, the heavy OWA, the heavy arithmetic weighted average and the heavy arithmetic OWA operator.

Keywords: Aggregation operators, Weighted average, OWA operator, Heavy aggregations.

1 INTRODUCTION

The aggregation operators (also known as aggregation functions) are becoming more relevant in the fuzzy community. Every year we see a lot of new developments and new types of aggregation operators. Some of the most common ones are the weighted average (WA) and the ordered weighted averaging (OWA) operator [14]. For further reading on different types of aggregation operators, see for example [1-5,10,12,15,18].

Recently, some authors [11-13] have tried to unify both concepts in the same formulation. It is worth noting the work developed by Torra [11] with the introduction of the weighted OWA (WOWA) operator and the work of Xu and Da [13] about the hybrid averaging (HA) operator. Both approaches arrived to a unified model between the OWA and the WA because both concepts were included in the formulation as particular cases. However, as it has been studied in [5-6], a better approach may be the use of a model that unifies both concepts and at the same time considers the degree of importance that each concept may have in the aggregation. This approach is known as the ordered weighted averaging – weighted averaging (OWAWA) operator [5-6]. Note that a similar approach has been suggested in [5,7] by using probabilities.

Another interesting aggregation approach introduced by Yager [16-17] is the heavy OWA (HOWA) operator. The main advantage of this model is that it allows the weighting vector to range between the usual OWA and the total operator. Thus, this approach is able to include a wide range of situations not considered by the usual OWA aggregation. For further information, see, for example [5,8-9,16-17].

The objective of this paper is to present the heavy OWAWA (HOWAWA) operator. It is a new aggregation operator that provides a unified model that uses OWAs, WAs, and heavy aggregations. Thus, we are able to provide a unified model between the OWA and the WA and at the same time we can allow their weighting vectors to move from the usual average to the total operator, that is, to the sum of all the arguments. One of the main advantages of this approach is that it includes a wide range of particular cases such as the heavy weighted average, the heavy OWA operator, different types of partial total operator, the arithmetic heavy weighted average, the arithmetic HOWA, the OWAWA operator, and a lot of other cases.

We also study the applicability of the new approach and we see that it is very broad because we can apply it in a lot of fields such as fuzzy set theory, statistics, engineering, decision theory, business and economics. We briefly develop a short illustrative example about the use of the HOWAWA operator.

The paper is organized as follows. In Section 2, we briefly review some basic concepts and the heavy weighted average. Section 3 presents the HOWAWA operator and

Section 4 studies some of its main families. In Section 5 we study the applicability of the new approach. Finally, in Section 6 we summarize the main conclusions of the paper.

2 PRELIMINARIES

2.1. THE OWA OPERATOR

The OWA operator [14] provides a parameterized family of aggregation operators between the maximum and the minimum. It can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$\text{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest of the a_i.

2.2. THE HEAVY OWA OPERATOR

The HOWA operator [16] is an extension of the OWA operator that allows the weighting vector to sum up to n. Thus, we are able to include the total operator in the aggregation. It can be defined as follows:

Definition 4. A HOWA operator is a mapping HOWA: $R^n \rightarrow R$ that has an associated weighting vector W with $w_j \in [0, 1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, such that:

$$\text{HOWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{2}$$

where b_j is the jth largest of the a_i and a_i is the argument variable.

As we can see, if $\sum_{i=1}^{n} w_i = 1$, then, we get the usual weighted average and if $\sum_{i=1}^{n} w_i = n$, the total operator.

2.3. THE HEAVY WEIGHTED AVERAGE

The heavy weighted average (HWA) is an extension of the weighted average for situations where we allow the weighting vector to move from the usual weighted average to the total operator. Thus, we get a more complete representation of the aggregation process. Note that the heavy aggregations are very useful for comparing the results obtained by using some kind of average and the sum of all the available results (or arguments), that is, the total results. For example, in distance measures [5,9] this is very useful because we can unify in the same formulation the normalized (or relative) distance with the absolute (or total) distance. The HWA can be defined as follows:

Definition 3. A HWA operator of dimension n is a mapping HWA: $R^n \rightarrow R$ that has an associated weighting vector W, with $w_i \in [0, 1]$ and $1 \leq \sum_{i=1}^{n} w_i \leq n$, such that:

$$\text{HWA}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i \tag{3}$$

where a_i represents the ith argument variable.

Note that if $\sum_{i=1}^{n} w_i = 1$, then, we get the usual weighted average and if $\sum_{i=1}^{n} w_i = n$, then, the total operator.

2.4. THE OWAWA OPERATOR

The ordered weighted averaging – weighted averaging (OWAWA) operator is an aggregation operator that unifies the WA and the OWA operator in the same formulation considering the degree that each concept has in the analysis [5-6]. It can be defined as follows.

Definition 4. An OWAWA operator of dimension n is a mapping OWAWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{OWAWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{4}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

By choosing a different manifestation in the weighting vector, we are able to obtain a wide range of particular types of OWAWA operators [5-6]. Especially, when $\beta = 0$, we get the WA, and if $\beta = 1$, we get the OWA operator. Other interesting cases are found when $w_j = 1/n$, for all a_i, because then, we get the arithmetic probability (AP). And if $v_i = 1/n$, for all a_i, we get the arithmetic OWA operator. Note that inside the arithmetic OWA we find the arithmetic maximum and minimum, and so on.

3 THE HEAVY OWAWA OPERATOR

The heavy ordered weighted averaging – weighted averaging (HOWAWA) operator is a new model that unifies the OWA operator and the weighted average in the same formulation. Therefore, both concepts can be seen as a particular case of a more general one. Moreover, we allow the aggregation to move between the minimum and the total operator. Thus, we can represent a lot of situations not included in the usual OWA and WA aggregations.

Note that some previous models already considered the possibility of using OWA operators and WAs in the same formulation. The main models are the weighted OWA (WOWA) operator [11-12] and the hybrid averaging (HA) operator [13]. In this case, we will get the heavy WOWA operator and the heavy HA operator. However, if we analyze in detail these models, we will see that these formulations cannot deal with heavy aggregations because then, the results and the process become inconsistent.

In the following, we are going to analyze the HOWAWA operator. It can be defined as follows.

Definition 5. A HOWAWA operator of dimension n is a mapping HOWAWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, according to the following formula:

$$\text{HOWAWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \quad (5)$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $1 \leq \sum_{i=1}^{n} v_i \leq n$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

Note that it is also possible to formulate the HOWAWA operator separating the part that strictly affects the HOWA operator and the part that affects the HWA. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models.

Definition 6. An HOWAWA operator is a mapping HOWAWA: $R^n \rightarrow R$ of dimension n, if it has an associated weighting vector W, with $1 \leq \sum_{j=1}^{n} w_j \leq n$ and $w_j \in [0, 1]$ and a weighting vector V that affects the WA, with $1 \leq \sum_{i=1}^{n} v_i \leq n$ and $v_i \in [0, 1]$, such that:

$$\text{HOWAWA}(a_1, \ldots, a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i \quad (6)$$

where b_j is the jth largest of the a_i and $\beta \in [0, 1]$.

In the following, we are going to give a simple example of how to aggregate with the HOWAWA operator. We consider the aggregation with both definitions.

Example 1. Assume the following arguments in an aggregation process: (60, 80, 30, 20, 40). Assume the following weighting vector W = (0.6, 0.6, 0.4, 0.2, 0.2) and the following probabilistic weighting vector V = (0.3, 0.3, 0.4, 0.5, 0.5). Note that the WA has a degree of importance of 50% while the weighting vector W of the OWA a degree of 50%. If we want to aggregate this information by using the HOWAWA operator, we will get the following. The aggregation can be solved either with (5) or (6). With (5) we calculate the new weighting vector as:

$$\hat{v}_1 = 0.5 \times 0.6 + 0.5 \times 0.3 = 0.45$$
$$\hat{v}_2 = 0.5 \times 0.6 + 0.5 \times 0.3 = 0.45$$
$$\hat{v}_3 = 0.5 \times 0.4 + 0.5 \times 0.5 = 0.45$$
$$\hat{v}_4 = 0.5 \times 0.2 + 0.5 \times 0.4 = 0.3$$
$$\hat{v}_5 = 0.5 \times 0.2 + 0.5 \times 0.5 = 0.35$$

Then, we calculate the aggregation process as follows:

HOWAWA = 0.45×80 + 0.45×60 + 0.45×40 + 0.3×30 + 0.35×20 = 97.

With (4), we aggregate as follows:

HOWAWA = 0.5 × (0.6 × 80 + 0.6 × 60 + 0.4 × 40 + 0.2 × 30 + 0.2 × 20) + 0.5 × (0.3 × 60 + 0.3 × 80 + 0.4 × 30 + 0.5 × 20 + 0.5 × 40) = 97.

Obviously, we get the same results with both methods.

Note that it is possible to distinguish between descending (DHOWAWA) and ascending (AHOWAWA) orders by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DHOWAWA and w^*_{n-j+1} the jth weight of the AHOWAWA operator.

If B is a vector corresponding to the ordered arguments b_j, we shall call this the ordered argument vector and W^T is the transpose of the weighting vector, then, the HOWAWA operator can be expressed as:

$$\text{HOWAWA}(a_1, \ldots, a_n) = W^T B \quad (7)$$

The HOWAWA is monotonic, bounded and idempotent. It is monotonic because if $a_i \geq u_i$, for all a_i, then, HOWAWA ($a_1, a_2,\ldots, a_n$) ≥ HOWAWA ($u_1, u_2\ldots, u_n$). It is bounded because the HOWAWA aggregation is delimitated by the minimum and the total operator. That is, Min{a_i} ≤ HOWAWA ($a_1, a_2,\ldots, a_n$) ≤ Total operator. It is idempotent because if $a_i = a$, for all a_i, then, HOWAWA ($a_1, a_2,\ldots, a_n$) = a. Note that this operator is not commutative because the weighted average is not commutative.

Another interesting issue to analyze are the measures for characterizing the weighting vector W. Following a similar methodology as it has been developed for the OWA operator [5,14,16] we can formulate the attitudinal character, the entropy of dispersion, the divergence of W and the balance operator. Note that these measures affect the weighting vector W but not the WAs because they are given as some kind of objective information that cannot be manipulated according to the decision makers interests. The first measure, the attitudinal character, can be defined as follows:

$$\alpha(W) = \frac{1}{|W|}\sum_{j=1}^{n}\left(\frac{n-j}{n-1}\right)w_j \quad (8)$$

As it can be seen, $\alpha(W) \in [0, 1]$. Note that the total operator has $\alpha(W) = 0.5$. The second measure, the entropy of dispersion, can be defined as:

$$H(W) = -\frac{1}{|W|}\sum_{j=1}^{n} w_j \ln\left(\frac{w_j}{|W|}\right) \quad (9)$$

Note that for the total operator, $H(W) = -\ln n$. A third measure that can be used based on [14], is the divergence of W, we will use:

$$Div(W) = \frac{1}{|W|}\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \quad (10)$$

If $|W| = n$, we get the divergence for the total operator and it is the same divergence than the average. That is, $Div(W) = (1/12)[(n + 1)/(n - 1)]$.

For the balance operator, we get:

$$BAL\,(W) = \frac{1}{|W|}\sum_{j=1}^{n}\left(\frac{n+1-2j}{n-1}\right)w_j \quad (11)$$

Note also that these four measures are reduced to the usual definitions [5,8-9,16-17] when $|W| = 1$.

4 FAMILIES OF HOWAWA OPERATORS

Different types of HOWAWA operators are found by using a different manifestation in the weighting vector or in the coefficient β. For example, we can obtain the following cases.

- OWAWA operator: When $\sum_{j=1}^{n} w_j = 1$ and $\sum_{i=1}^{n} v_i = 1$.
  - The particular cases of the OWAWA operator are explained in [10].
- Total operator: When $\sum_{j=1}^{n} w_j = n$ and $\sum_{i=1}^{n} v_i = n$.
- The total weighted average operator: When $\sum_{j=1}^{n} w_j = n$ and $\sum_{i=1}^{n} v_i = 1$.
- The total heavy weighted average operator: When $\sum_{j=1}^{n} w_j = n$.
- The total OWA operator: When $\sum_{i=1}^{n} v_i = n$ and $\sum_{j=1}^{n} w_j = 1$.
- The total heavy OWA operator: When $\sum_{i=1}^{n} v_i = n$.
- The weighted heavy OWA operator: When $\sum_{i=1}^{n} v_i = 1$.
- The OWA heavy weighted average: When $\sum_{j=1}^{n} w_j = 1$.
- The HOWA operator: If $\beta = 1$.
- The heavy weighted average: If $\beta = 0$.
- The heavy average: If $w_j = |W|/n$ and $v_i = |W|/n$, or $w_j = |W|/n$ and $\beta = 1$, or $v_i = |W|/n$ and $\beta = 0$.
- The heavy arithmetic weighted average: If $w_j = |W|/n$.
- The heavy arithmetic OWA operator: If $v_i = |W|/n$.
- The arithmetic heavy weighted average: If $w_j = 1/n$.
- The arithmetic heavy OWA operator: If $v_i = 1/n$.
- The arithmetic mean: If $w_j = 1/n$ and $v_i = 1/n$, for all i.
- The minimum: When $w_n = 1$, $w_j = 0$, for all $j \neq n$ and $\beta = 1$.
- The heavy weighted minimum: When $w_n = 1$, $w_j = 0$, for all $j \neq n$.
- Push up allocation: We use $w_j = (1 \wedge (|W| - (j - 1))) \vee 0$ and $\beta = 1$.
- Heavy weighted push up allocation: We use $w_j = (1 \wedge (|W| - (j - 1))) \vee 0$.
- Weighted push up allocation: In this case, $w_j = (1 \wedge (|W| - (j - 1))) \vee 0$ and $\sum_{i=1}^{n} v_i = 1$.
- The maximum: When $w_1 = 1$, $w_j = 0$, for all $j \neq 1$ and $\beta = 1$.
- Push down allocation: We use $w_{n-j+1} = (1 \wedge (|W| - (j - 1))) \vee 0$ and $\beta = 1$.
- Heavy weighted push down allocation: In this case, $w_{n-j+1} = (1 \wedge (|W| - (j - 1))) \vee 0$.
- Weighted push down allocation: We use $w_{n-j+1} = (1 \wedge (|W| - (j - 1))) \vee 0$ and $\sum_{i=1}^{n} v_i = 1$.

- Median type allocation: If n is even, we allocate the weights for j = 1 to a as $w_{a+j} = w_{a+1-j} = [1 \wedge ((|W| - 2(j - 1))/2)] \vee 0$. If n is odd, we allocate the weights for j = 1 to a as $w_{a+1} = 1$ and $w_{a+1-j} = w_{a+1+j} = [1 \wedge ([(|W| - 1) - 2(j - 1)]/2)] \vee 0$.
- Heavy weighted median allocation: The same than the median type allocation but $\beta \neq 1$.
- Weighted median allocation: The same than the heavy weighted median allocation and $\sum_{i=1}^{n} v_i = 1$.
- Step-HOWAWA allocation: Assuming β = 1 and b = Min[(K – 1), (n – K)], we allocate the weights for j to b as $w_K = 1$ and $w_{K+j} = w_{K-j} = [1 \wedge ([(|W| - 1) - 2(j - 1)]/2)] \vee 0$.
- Heavy weighted step-HOWAWA allocation: We assume b = Min[(K – 1), (n – K)], and we allocate the weights for j to b as $w_K = 1$ and $w_{K+j} = w_{K-j} = [1 \wedge ([(|W| - 1) - 2(j - 1)]/2)] \vee 0$.
- Weighted step-HOWAWA allocation: The same than the heavy weighted step-HOWAWA allocation and $\sum_{i=1}^{n} v_i = 1$.
- Olympic-average allocation: We have to distinguish between two cases. Note that in both cases we have β = 1.
    - In the first case, where $|W| < n - 2m$, we allocate the weight as $w_j = |W|/(n - 2m)$ for $j = m + 1$ to $n - m$, and $w_j = 0$ for j = 1 to m and for $j = n - m + 1$ to n.
    - In the second case, where $|W| \geq n - 2m$, we allocate the weights as $w_j = 1$ for $j = m + 1$ to $n - m$ and $w_{m+1-j} = w_{n-m+j} = [1 \wedge ([(|W| - (n - 2m)) - 2(j - 1)]/2)] \vee 0$ for j = 1 to m.
- Heavy weighted olympic allocation: The same than the olympic-average allocation but $\beta \neq 1$.
- Weighted olympic allocation: The same than the heavy weighted olympic allocation and $\sum_{i=1}^{n} v_i = 1$.
- Arrow-Hurwicz allocation: Assuming that β = 1, $|W| = q$ and dimension n, we define the weights in two directions, push up and push down. First, we calculate $\omega_j = (1 \wedge (\lambda q - (j - 1))) \vee 0$ for j = 1 to n and $\hat{w}_{n-j+1} = (1 \wedge ((1 - \lambda)q - (j - 1))) \vee 0$ for j = 1 to n. Then, we define the weights as $w_i = \omega_i + \hat{w}_i$. Note that $\omega_j = 0$ for all $j \geq \lambda q + 1 \geq \lambda|W| + 1$ and $\hat{w}_j = 0$ for $j \leq n - (1 - \lambda)q \leq n - |W| + \lambda|W|$.
- The heavy weighted Arrow-Hurwicz allocation: The same than the Arrow-Hurwicz allocation but $\beta \neq 1$.
- The weighted Arrow-Hurwicz allocation: The same than the heavy weighted olympic allocation and $\sum_{i=1}^{n} v_i = 1$.

Note that it is possible to develop other types of HOWAWA operators by using other manifestations of the weighting vector of the WA and the OWA.

5 APPLICABILITY OF THE HOWAWA

The HOWAWA operator can be used in a lot of applications. In general, we can conclude that the HOWAWA operator can be implemented in all the studies that use the weighted average or the OWA operator because both of them are particular cases of this new approach. Therefore, all the studies that use the OWA or the WA can be revised and extended with this new approach. For example, we can apply it in fuzzy set theory, soft computing, statistics, decision theory, operational research, business and economics, politics, engineering and a lot of other sciences such as mathematics, physics, chemistry and biology.

In the following we are going to develop a simple numerical example. Assume we have three experts that are analyzing the benefits of a company for the next year. Expert 1 knows very well the European market of the company, Expert 2 the American market and Expert 3 the Asian market. Each of them has confidential information about the market they are working on. Therefore, when predicting the results, each of them gives a general result that takes into account this confidential information. In this example we assume that the three experts have a degree of 50% of independence in their information. Therefore, if we calculate the average result in order to see the predicted benefits by the experts, we have to allow the weighting vector to sum up to 1.5 so we take into account all the information. In this example we assume that all the experts provide information that it is equally important. Therefore, we use the following weighting vector V = (0.5, 0.5, 0.5). The company is very pessimistic so they want to use an OWA aggregation that reflects this aspect in the following way: W = (0.2, 0.2, 0.6).Note that this weighting vector has to be increased 50% obtaining W^* = (0.3, 0.3, 0.9). Assume that the experts predict the following results:

- Expert 1: 300 millions €.
- Expert 2: 250 millions €.
- Expert 3: 350 millions €.

If we aggregate this information using a heavy aggregation we get the following. First, we have to mix the weights of the OWA and the WA in order to provide a unified aggregation. In this example, we assume that both concepts have a degree of importance of 50%.

$$\hat{v}_1 = 0.5 \times 0.3 + 0.5 \times 0.5 = 0.4$$
$$\hat{v}_2 = 0.5 \times 0.3 + 0.5 \times 0.5 = 0.4$$
$$\hat{v}_3 = 0.5 \times 0.9 + 0.5 \times 0.5 = 0.7$$

HOWAWA = 0.4 × 350 + 0.4 × 300 + 0.7 × 250 = 435.

As we can see, assuming the previous conditions, the company would assume that the predicted benefit for the next period is 435 millions €.

5 CONCLUSIONS

We have presented a new aggregation operator that uses the weighted average and the OWA operator in the same formulation and considering the degree of importance that each concept has in the aggregation. Moreover, ths new approach allows the weighting vector of the OWA and the WA to range from the usual average to the total operator. Therefore, we have been able to consider a lot of new aggregation operators. We have called this new approach the HOWAWA operator. We have studied some of its main particular cases such as the OWAWA operator, the total operator, the total weighted average operator, the total heavy weighted average operator, the arithmetic heavy weighted average, the arithmetic heavy OWA operator, and a lot of other cases.

We have also studied the applicability of the new approach and we have seen that it is very broad because all the studies that use the OWA or the WA can be revised and extended with this new approach.

In future research, we will develop further extensions of this approach by using other sources of information such as the use of interval numbers, fuzzy numbers and expertons. We will also consider the use of order-inducing variables and distance measures and we will develop some applications in decision making problems.

Acknowledgements

We would like to thank the anonymous referees for valuable comments that have improved the quality of the paper.

References

[1] G. Beliakov, A. Pradera and T. Calvo, *Aggregation Functions: A Guide for Practitioners*. Berlin: Springer-Verlag, 2007.

[2] V. Cutello and J. Montero, Hierarchical aggregation of OWA operators: basic measures and related computational problems. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 3:17-26, 1995.

[3] V. Cutello and J. Montero, Recursive connective rules. *International Journal of Intelligent Systems*, 14:3-20, 1999.

[4] D. Gómez and J. Montero, A discussion on aggregation operators. *Kybernetika*, 40:107-120, 2004.

[5] J.M. Merigó, *New extensions to the OWA operators and its application in decision making* (In Spanish). PhD Thesis, Department of Business Administration, University of Barcelona, 2008.

[6] J.M. Merigó, On the use of the OWA operator in the weighted average and its application in decision making. In: *Proceedings of the WCE 2009 Conference*, London, United Kingdom, pp. 82-87, 2009.

[7] J.M. Merigó, Probabilistic decision making with the OWA operator and its application in investment management. In: *Proceedings of the IFSA-EUSFLAT Conference*, Lisbon, Portugal, pp. 1364-1369, 2009.

[8] J.M. Merigó and M. Casanovas, Using fuzzy numbers in heavy aggregation operators. *International Journal of Information Technology*, 4**,** 177-182, 2008.

[9] J.M. Merigó and M. Casanovas, Distance measures in induced and heavy aggregation operators. In: *Proceedings of the EUROFUSE Conference*, Pamplona, Spain, pp. 219-226, 2009.

[10] J.M. Merigó and A.M. Gil-Lafuente, The induced generalized OWA operator. *Information Sciences*, 179:729-741, 2009.

[11] V. Torra, The weighted OWA operator. *International Journal of Intelligent Systems*, 12:153-166, 1997.

[12] V. Torra and Y. Narukawa, *Modeling Decisions: Information Fusion and Aggregation Operators*. Berlin: Springer-Verlag, 2007.

[13] Z.S. Xu and Q.L. Da, An overview of operators for aggregating information. *International Journal of Intelligent Systems*, 18:953-968, 2003.

[14] R.R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems, Man and Cybernetics B*, 18:183-190, 1988.

[15] R.R. Yager, Families of OWA operators. *Fuzzy Sets and Systems*, 59:125-148, 1993.

[16] R.R. Yager, Heavy OWA operators. *Fuzzy Optimization and Decision Making*, 1:379-397, 2002.

[17] R.R. Yager, Monitored heavy fuzzy measures and their role in decision making under uncertainty. *Fuzzy Sets and Systems*, 139:491-513, 2003.

[18] R.R. Yager and J. Kacprzyk, *The Ordered Weighted Averaging Operators: Theory and Applications*. Norwell: Kluwer Academic Publishers, 1997.