DECISION MAKING WITH DISTANCE MEASURES, WEIGHTED AVERAGES AND INDUCED OWA OPERATORS

José M. Merigó **Anna M. Gil-Lafuente**

Department of Business Administration, University of Barcelona,
Av. Diagonal 690, 08034 Barcelona, Spain, {jmerigo, amgil}@ub.edu

Abstract

We develop a new decision making model by using distance measures, weighted averages and OWA operators. We introduce the induced ordered weighted averaging – weighted averaging distance (IOWAWAD) operator. We study some of its main properties and particular cases such as the weighted Hamming distance, the induced OWA distance (IOWAD), the arithmetic weighted distance and the arithmetic IOWAD operator. We apply the new approach in a decision making problem about product management.

Keywords: Decision making, Distance measures, Induced OWA operator, Weighted average.

1 INTRODUCTION

In the literature, we find a wide range of aggregation operators (or aggregation techniques) that have been useful in the field of decision making [2,5-11,15,17-20]. A very useful technique for doing this is the Hamming distance [3] and more generally all the distance measures [2-5,11]. The main advantage of using distance measures in decision making is that we can compare the alternatives of the problem with some ideal result [2-5,11].

In order to develop the decision process we may use different types of aggregation operators (also known as aggregation functions) for the normalization process of the distance measures. The weighted average (WA) is one of the most common aggregation operators found in the literature. It can be used in a wide range of problems including statistics, economics and engineering. Another interesting aggregation operator is the ordered weighted averaging (OWA) operator [15]. The OWA operator provides a parameterized family of aggregation operators that range from the maximum to the minimum, see, for example [1,5-20].

Recently, some authors [12-13,14] have tried to unify both concepts in the same formulation such as the work developed by Torra [12-13] with the introduction of the weighted OWA (WOWA) operator and the work of Xu and Da [14] about the hybrid averaging (HA) operator. Both models unified the OWA and the WA because both concepts were included in the formulation as particular cases. However, as it has been studied in [5-7], these models seem to be a partial unification but not a total one because they can unify them but they cannot consider how relevant these concepts are in the specific problem considered. For example, in some problems we may prefer to give more importance to the OWA operator because we believe that it is more relevant and vice versa. Recently, Merigó [5-6] has developed a new approach that unifies the OWA and the WA in the same formulation considering the degree of importance that each concept may have in the problem. He called it the ordered weighted averaging – weighted averaging (OWAWA) operator. Moreover, he has further extended this approach by using order-inducing variables and uncertain information [5,7].

The aim of this paper is to develop a new distance measure that we call the induced OWAWA distance operator and apply it in a decision making problem about product management. The IOWAWAD operator is a distance aggregation operator that provides a parameterized family of aggregation operators between the maximum and the minimum distances. It normalizes the Hamming distance (or other distances) with the IOWAWA operator. Therefore, we are able to include the WA and the IOWA operator at the same time in the Hamming distance. Thus, it includes a wide range of distances such as the arithmetic weighted Hamming distance, the arithmetic IOWAD operator, the OWAWA distance operator, the IOWAD operator, the weighted Hamming distance, the maximum weighted Hamming distance, and a lot of other particular cases.

This paper is structured as follows. In Section 2, we review some basic concepts. Section 3 presents the IOWAWAD operator and Section 4 gives a numerical example. In Section 5 we summarize the conclusions.

2 PRELIMINARIES

2.1. THE HAMMING DISTANCE

The Hamming distance [3] is a very useful technique for calculating the differences between two elements, two sets, etc. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets and intuitionistic fuzzy sets. For two sets A and B, it can be defined as follows.

Definition 1. A weighted Hamming distance of dimension n is a mapping d_{WH}: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$. Then:

$$d_{WH}(A, B) = \left(\sum_{i=1}^{n} w_i \mid a_i - b_i \mid \right) \tag{1}$$

where a_i and b_i are the ith arguments of the sets A and B respectively.

Note that it is possible to generalize this definition to all the real numbers by using $R^n \times R^n \rightarrow R$. Note also that if $w_i = 1/n$, for all i, then, we get the normalized Hamming distance. For the formulation used in fuzzy set theory, see for example [2-5,11].

2.2. THE OWA OPERATOR

The OWA operator [15] provides with a parameterized family of aggregation operators that include the arithmetic mean, the maximum and the minimum. It can be defined as follows.

Definition 2. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$OWA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{2}$$

where b_j is the jth largest of the a_i.

2.3. THE INDUCED OWA OPERATOR

The IOWA operator [17,19] is an extension of the OWA operator. Its main difference is that the reordering step is not carried out with the values of the arguments a_i. In this case, the reordering step is developed with order inducing variables that reflects a more complex reordering process. It can be defined as follows.

Definition 3. An IOWA operator of dimension n is a mapping IOWA: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, then:

$$\mathrm{IOWA}(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{3}$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

2.4. THE INDUCED OWA DISTANCE OPERATOR

The induced OWAD (IOWAD) operator [5] represents an extension of the traditional normalized distance by using IOWA operators. The difference is that we reorder the arguments of the individual distances with order inducing variables that represents a complex reordering process. It can be defined as follows.

Definition 4. An induced OWAD operator of dimension n is a mapping IOWAD: $R^n \times R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$. Then, the distance between two sets is:

$$IOWAD(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j D_j \tag{4}$$

where D_j is the $|x_i - y_i|$ value of the IOWAD triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, x_i is the ith characteristic of the ideal *set*, y_i is the ith characteristic of the kth alternative and $k = 1, 2, \ldots, m$.

2.5. THE OWAWA OPERATOR

The ordered weighted averaging – weighted averaging (OWAWA) operator is an aggregation operator that unifies the WA and the OWA operator in the same formulation considering the degree of importance that each concept has in the analysis [5]. It is defined as follows.

Definition 5. An OWAWA operator of dimension n is a mapping OWAWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{5}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

By choosing a different manifestation in the weighting vector, we are able to obtain a wide range of particular types of OWAWA operators [5]. Especially, when $\beta = 0$, we get the WA, and if $\beta = 1$, we get the OWA operator.

2.6. THE INDUCED OWAWA OPERATOR

The induced ordered weighted averaging – weighted averaging (OWAWA) operator is a new model that unifies the OWA operator and the weighted average in the same formulation and considering a complex reordering process based on order inducing variables. It can be defined as follows.

Definition 6. An IOWAWA operator of dimension n is a mapping IOWAWA: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{IOWAWA}\ (\langle u_1, a_1\rangle, \langle u_2, a_2\rangle \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{6}$$

where b_j is the a_i value of the IOWAWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

3 THE INDUCED OWAWA DISTANCE

The induced ordered weighted averaging weighted averaging distance (IOWAWAD) operator is a distance measure that uses the WA and the OWA operator in the normalization process of the Hamming distance by using the IOWAWA operator. Then, the reordering of the individual distances is developed according to order-inducing variables that represent a complex reordering process of the individual distances formed by comparing two sets. The main advantage of this new approach is that it is able to deal with situations where we have some subjective information about the possibility of occurrence of the different results and the attitudinal character of the decision maker that it is assessed with order inducing variables in order to represent his attitude in a complete way considering a wide range of aspects such as the degree of optimism, psychological aspects and time pressure. It can be defined as follows for two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Definition 7. An IOWAWAD operator of dimension n is a mapping IOWAWAD: $R^n \times R^n \times R^n \rightarrow R$ that has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$IOWAWAD\ (\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{7}$$

where b_j is the $|x_i - y_i|$ value of the IOWAWAD triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, x_i is the ith argument of the set $X = \{x_1, \ldots, x_n\}$, y_i is the ith argument of the set $Y = \{y_1, \ldots, y_n\}$, each argument or individual distance $|x_i - y_i|$ has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to the jth largest u_i.

Note that it is also possible to formulate the IOWAWAD operator separating the part that strictly affects the IOWAD operator and the part that affects the WAD.

Definition 8. An IOWAWAD operator is a mapping *IOWAWAD*: $R^n \times R^n \times R^n \rightarrow R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$ and a weighting vector V that affects the WAD, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$IOWAWAD\ (\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \\ = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i |x_i - y_i| \tag{8}$$

where b_j is the $|x_i - y_i|$ value of the IOWAWAD triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, x_i is the ith argument of the set $X = \{x_1, \ldots, x_n\}$, y_i is the ith argument of the set $Y = \{y_1, \ldots, y_n\}$, and $\beta \in [0, 1]$.

Note that if the weighting vector is not normalized, i.e., $\hat{V} = \sum_{j=1}^{n} \hat{v}_j \neq 1$, then, the IOWAWAD operator can be expressed as:

$$IOWAWAD\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \frac{1}{\hat{V}} \sum_{j=1}^{n} \hat{v}_j b_j \quad (9)$$

If D is a vector corresponding to the ordered arguments b_j, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector, then, the IOWAWAD operator can be represented as follows:

$$IOWAWAD\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = W^T D \quad (10)$$

Note that it is possible to distinguish between descending (DIOWAWAD) and ascending (AIOWAWAD) orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DIOWAWAD and w^*_{n-j+1} the jth weight of the AIOWAWAD operator.

Note that IOWAWAD $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = 0$ if and only if $x_i = y_i$ for all $i \in [1, n]$. Note also that IOWAWAD $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) =$ IOWAWAD $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$.

The IOWAWAD operator is monotonic, bounded and idempotent. It is monotonic because if $|x_i - y_i| \geq |s_i - t_i|$, for all $|x_i - y_i|$, then, $IOWAWAD\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq IOWAWAD\,(\langle u_1, s_1, t_1\rangle, \ldots, \langle u_n, s_n, t_n\rangle)$. It is bounded because the IOWAWAD aggregation is delimitated by the minimum and the maximum. That is, $\text{Min}\{|x_i - y_i|\} \leq IOWAWAD\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \text{Max}\{|x_i - y_i|\}$. It is idempotent because if $|x_i - y_i| = |x - y|$, for all $|x_i - y_i|$, then, $IOWAWAD\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = |x - y|$.

Note that we could study different measures for characterizing the weights, especially the OWA part, such as the entropy of dispersion, the divergence of W or the balance operator. The entropy of dispersion is defined as follows:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (11)$$

For the balance operator, we get:

$$BAL\,(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1}\right) w_j \quad (12)$$

And for the divergence of W:

$$DIV(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \quad (13)$$

A further interesting issue to consider is the different families of IOWAWAD operators that are found in the weighting vector W and the coefficient β.

- If $\beta = 0$, we get the weighted Hamming distance.
- If $\beta = 1$, we get the IOWAD operator.
- The arithmetic weighted distance (if $w_j = 1/n$, for all j).
- The arithmetic IOWA distance (if $v_i = 1/n$, for all i).
- The normalized Hamming distance (if $v_i = 1/n$, for all i, and $w_j = 1/n$, for all j).
- The maximum weighted distance ($w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\{|x_i - y_i|\}$).
- The minimum weighted distance ($w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\{|x_i - y_i|\}$).
- The IOWAWA operator (if one of the sets is empty).
  - The OWAWA (the ordered position of the u_i is the same than the ordered position b_j).
  - The IOWA ($\beta = 1$).
  - The OWA ($\beta = 1$ and the ordered position of the u_i is the same than the ordered position b_j).
  - The weighted average ($\beta = 0$).
- The Hurwicz weighted distance criteria ($w_p = \alpha$, with $u_p = \text{Max}\{|x_i - y_i|\}$; $w_q = 1 - \alpha$, $u_q = \text{Min}\{|x_i - y_i|\}$; and $w_j = 0$, for all $j \neq p, q$).
- The step-IOWAWAD ($w_k = 1$ and $w_j = 0$, for all $j \neq k$).
- The olympic-IOWAWAD operator ($w_1 = w_n = 0$, and $w_j = 1/(n-2)$ for all others).
- The general olympic-IOWAWAD operator ($w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others $w_{j*} = 1/(n-2k)$, where $k < n/2$).
- The S-IOWAWAD ($w_1 = (1/n)(1 - (\alpha + \beta) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta)$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$).
- The centered-IOWAWAD (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).
- Etc.

Note that other families of IOWAWAD operators may be used following a similar methodology as it has been developed for the OWA operator and its extensions [1,5-20]. Moreover, we could extend this analysis to other types of distances such as the Euclidean (or quadratic) distance, the Minkowski (or generalized) distance and the quasi-arithmetic distance.

It is worth noting that some previous models already considered the possibility of using OWA operators and WAs in the same formulation. The main models are the WOWA operator [12-13] and the HA operator [14]. In this case, following these methodologies, we could develop the induced WOWA distance (IWOWAD) operator and the induced hybrid averaging distance (IHAD) operator in a similar way as it has been done in the IOWAWAD operator. Another approach that could be analyzed is the concept of immediate probability [18] because we could use it as a weighted average (immediate weighted average). Thus, we could also develop the induced immediate weighted averaging distance (IIWAD) operator. Note that other models could also be considered under this framework where we are developing different extensions and generalizations. However, we believe that the approach that we are following seems to be the most complete. Therefore, we will not analyze in detail the other approaches although sometimes we will make some remarks about them.

4 ILLUSTRATIVE EXAMPLE

The IOWAWAD operator can be implemented in a wide range of applications where it is possible to use weighted averages, distance measures and OWA operators. Thus, we see that the applicability is incredibly broad because all the previous models and theories that use the WA can be revised and extended by using the IOWAWAD.

Summarizing some of the main fields where it is possible to apply the IOWAWAD operator, we can mention statistical theory, mathematics, economics, decision theory, engineering, soft computing and physics.

We focus on an application in decision making about selection of products. We analyze a customer that wants to buy a new product and he has five different brands that give him more or less the product he wants.

- A_1 = Product A.
- A_2 = Product B.
- A_3 = Product C.
- A_4 = Product D.
- A_5 = Product E.

In order to evaluate these products, the customer analyzes the products considering five main characteristics (or attributes):

- C_1 = Prize of the product.
- C_2 = Quality of the product.
- C_3 = Knowledge about the brand of the product.
- C_4 = Intuitive preference of the customer.
- C_5 = Other variables.

The results of the available products, depending on the characteristic C_i and the alternative A_k that the decision maker chooses, are shown in Table 1.

Table 1: Available products

	C_1	C_2	C_3	C_4	C_5
A_1	0.6	0.3	0.4	0.8	0.7
A_2	0.7	0.9	0.2	0.3	0.4
A_3	0.4	0.7	0.3	0.5	0.6
A_4	0.8	0.2	0.1	0.6	0.9
A_5	0.4	0.7	0.6	0.3	0.7

According to the objectives of the decision maker, he establishes the following ideal product. The results are shown in Table 2.

Table 2. Ideal product.

	C_1	C_2	C_3	C_4	C_5
I	0.9	1	0.8	0.8	0.9

Due to the fact that the attitudinal character is very complex because it involves the opinion of different members of the board of directors, the recruiters use order inducing variables to express it. The results are represented in Table 3.

Table 3: Order inducing variables

	C_1	C_2	C_3	C_4	C_5
U	14	26	22	17	19

With this information, it is possible to develop different methods based on the IOWAWAD operator for selecting a product. In this example, we will consider the NHD, the WHD, the IOWAD and the IOWAWAD operator. We will assume that β = 0.5 and the following weights: W = (0.1, 0.1, 0.2, 0.2, 0.4) and V = (0.3, 0.3, 0.2, 0.1, 0.1). The results are shown in Table 4.

Table 4: Aggregated results

	NHD	WHD	IOWAD	IOWAWAD
A_1	0.32	0.4	0.27	0.335
A_2	0.38	0.31	0.35	0.33
A_3	0.38	0.4	0.22	0.31
A_4	0.36	0.43	0.23	0.33
A_5	0.34	0.35	0.39	0.37

If we establish an ordering of the products, a typical situation if we want to consider more than one alternative, then, we get the results shown in Table 5. Note that the first alternative in each ordering is the optimal choice.

Table 5: Ordering of the products

	Ordering
NHD	$A_1 \succ A_5 \succ A_4 \succ A_2 = A_3$
WHD	$A_2 \succ A_5 \succ A_1 = A_3 \succ A_4$
IOWAD	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
IOWAWAD	$A_3 \succ A_2 = A_4 \succ A_1 \succ A_5$

As we can see, depending on the aggregation operator used, the ordering of the products may lead to different decisions.

5 CONCLUSIONS

We have presented the IOWAWAD operator. It is a new aggregation operator that uses a unified model between the WA and the OWA. Moreover, it uses distance measures in the analysis and a reordering process based on order-inducing variables. We have studied some of its main properties and we have found a lot of particular cases such as the normalized Hamming distance, the weighted Hamming distance, the OWAD, the IOWAD, the IOWAWA operator, the maximum weighted distance, the minimum weighted distance, the arithmetic weighted distance and the arithmetic IOWA distance.

We have also analyzed the applicability of this approach because we have seen that we can apply it in all the studies that use distance measures, weighted averages or OWAs. We have focussed on a decision making problem about product management.

In future research, we expect to develop further developments by adding new characteristics in the problem such as the use of other types of distance measures such as the Euclidean distance, the Minkowski distance or the quasi-arithmetic distance. Moreover, we will also analyze the use of probabilistic information and uncertain environments.

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