AGGREGATION AND ARITHMETIC OPERATIONS ON FUZZY NATURAL NUMBER-VALUED MULTISETS

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Abstract

In this paper we propose the extension of some aggregation or arithmetic operations between discrete fuzzy numbers to discrete fuzzy number-valued multisets such as the addition, the union and the intersection and, even, the order. For this reason, we prove or recall some properties of discrete fuzzy numbers and we undertake their study on multisets.

Keywords: Aggregation operators, Multisets, discrete fuzzy number, lattice, monoid.

1 INTRODUCTION

A (crisp) *multiset* over a set of *types* X is a mapping $M : X \to \mathbb{N}$. A survey of the mathematics of multisets, including their axiomatic foundation, can be found in [1]. In [2], order and triangular operations between multisets are studied. Multisets are also called *bags* in the literature [17].

According to the usual interpretation of a multiset $M : X \to \mathbb{N}$, it describes a set or *universe*, Ω, which consists of $M(x)$ “exact” copies of each type $x \in X$, without specifying which element of the universe is a copy of which element of X. The number $M(x)$ is usually called the *multiplicity* of x in the multiset M. Notice, in particular, that the set or universe described by the multiset does not contain any element that is not a copy of some $x \in X$, and that an element of it cannot be a copy of two different types. So, each type $x \in X$ defines a subset Ω_x of the universe constituted by their ”exact” copies and we have $\Omega_x \cap \Omega_y = \emptyset$, if $x \neq y$ and $\Omega = \bigcup_{x\in X} \Omega_x$. The *multiplicity*, $M(x)$, of x in the multiset M is the *cardinal* of the subset Ω_x.

In [2], the authors introduced a more general definition of ”extended multiset” as mappings $M : X \to L$, where L is a finite or infinite chain of natural numbers, or, even, it can be $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, with the usual operations and order. This definition allows to extend several aggregation operators defined in L, such as t-norms or t-conorms, to multisets.

A natural generalization of this interpretation of multisets leads to the notion of *multisets with fuzzy values* [11, 12] over a set of *types* X. Such a multiset describes for each $x \in X$, a set consisting of ”possibly inexact” copies of x with a degree of similarity valued in [0,1]. In this way, in [12] an immediate generalization of crisp multisets using fuzzy numbers instead of natural numbers is proposed. So, provided a suitable definition of fuzzy number (triangular, trapezoidal, Gauss-shaped, etc [10]), it is possible to consider fuzzy Number-Valued multisets defined over X.

In this paper we impose two restrictions on this interpretation of a fuzzy multiset, parallel to those highlighted in the crisp case, that allows us to slightly modify this definition. First, we assume that if an element of the set is an inexact copy of x with a degree of similarity $t > 0$, then it cannot be an inexact copy of any other type in X with a non-negative degree of similarity. And second, the set or universe described by the fuzzy multiset does not contain any element that is not a copy of some $x \in X$ with some non-negative degree of similarity. These two conditions entail that each type $x \in X$ defines a fuzzy subset Ω_x of the universe Ω constituted by all the copies of x, each one with its degree of similarity with x and we have $\Omega_x \cap \Omega_y = \emptyset$, if $x \neq y$ and $\Omega = \bigcup_{x\in X} \Omega_x$, i.e. $\Omega_x(w) \wedge \Omega_y(w) = 0$ and $\bigvee_{x\in X} \Omega_x(w) = 1$ for all $w \in \Omega$.

In order to define a ”multiplicity” or *fuzzy multiplicity* of each type for a *fuzzy multiset* over X, we need to associate to each $x \in X$ the *cardinality* of the fuzzy set Ω_x. The problem of “counting” fuzzy sets has generated a lot of literature since Zadeh’s first definition of the cardinality of fuzzy sets [9, 10]. In particular, the scalar cardinalities of fuzzy sets, which associate to each fuzzy set a positive real number, have been studied from the axiomatic point of view [8, 16] with the aim of capturing different ways of counting additive aspects of fuzzy sets like the cardinalities of supports, of levels, of cores, etc. In a similar way, the fuzzy cardinalities of fuzzy sets [7, 9, 16], which associate to any fuzzy set a convex fuzzy natural number, have also been studied from the axiomatic point of view.

In this paper, taking into account that the fuzzy cardinality of a fuzzy set is a fuzzy natural number,ie, a discrete fuzzy number whose support is a subset of consecutive natural numbers, we consider a *Fuzzy Natural Number-Valued multiset* defined over a set X as a mapping $M : X \rightarrow FNN$ where FNN is the set of discrete fuzzy numbers with support a subset of consecutive natural numbers.

In order to define some operations in this framework, the first step is to recall the operations between fuzzy natural numbers. If we know that the set of fuzzy natural numbers is closed with respect to an operation, $\overline{O}$, we can define an operation between multisets functionally: $\overline{O}(M, N)(x) = \overline{O}(M(x), N(x)), \forall x \in X$. Thus, we study the addition, the union and the intersection as functionally defined extensions of operations between fuzzy natural numbers.

2 PRELIMINARIES

2.1 Multisets

Let X be a crisp set. A (*crisp*) *multiset* over X is a mapping $M : X \rightarrow \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers including the 0. A multiset M over X is *finite* if its *support*

$$supp(M) = \{x \in X | M(x) > 0\}$$

is a finite subset of X. We shall denote the sets of all multisets and of all finite multisets over a set X by $MS(X)$ and $FMS(X)$, respectively, and by $\perp$ the *null multiset*, defined by $\perp(x) = 0$ for each $x \in X$.

For every $A, B \in MS(X)$, their *sum* [13] $A + B$ is the multiset defined pointwise by

$$(A + B)(x) = A(x) + B(x), \quad x \in X.$$

Let us mention here that it has been argued that this sum $+$, also called *additive union*, is the right notion of union of multisets. According to the interpretation of multisets as sets of copies of types explained in the introduction, this sum corresponds to the disjoint union of sets, as it interprets that all copies of each x in the set represented by A are different from all copies of it in the set represented by B. This additive sum has quite different properties from the ordinary union of sets. For instance, the collection of submultisets of a given multiset is not closed under this operation and consequently no sensible notion of complement within this collection exists.

For every $A, B \in MS(X)$, their *join* $A \vee B$ and *meet* $A \wedge B$ are respectively the multisets over X defined pointwise by $(A \vee B)(x) = \max(A(x), B(x))$ and $(A \wedge B)(x) = \min(A(x), B(x))$, $x \in X$. If A and B are finite, then $A + B$, $A \vee B$ and $A \wedge B$ are also finite. A partial order $\leq$ on $MS(X)$ is defined by $A \leq B$ if and only if $A(x) \leq B(x)$ for every $x \in X$. If $A \leq B$, then their *difference* $B - A$ is the multiset defined pointwise by

$$(B - A)(x) = B(x) - A(x).$$

2.2 Discrete Fuzzy Numbers

By a fuzzy subset of the set of real numbers, we mean a function $u : \mathbb{R} \rightarrow [0, 1]$. For each fuzzy subset u, let $u^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$ be its α-level set (or α-cut). By $supp(u)$, we mean the support of u, i.e. the set $\{x \in \mathbb{R} : u(x) > 0\}$. By u^0, we mean the closure of $supp(u)$.

Definition 2.1 *[14] A fuzzy subset u of $\mathbb{R}$ with membership mapping $u : \mathbb{R} \rightarrow [0, 1]$ is called* discrete fuzzy number *if its support is finite, i.e., there are $x_1, ..., x_n \in \mathbb{R}$ with $x_1 < x_2 < ... < x_n$ such that $supp(u) = \{x_1, ..., x_n\}$, and there are natural numbers s, t with $1 \leq s \leq t \leq n$ such that:*

1. *$u(x_i)$=1 for any natural number i with $s \leq i \leq t$ (*core*)*
2. *$u(x_i) \leq u(x_j)$ for each natural number i, j with $1 \leq i \leq j \leq s$*
3. *$u(x_i) \geq u(x_j)$ for each natural number i, j with $t \leq i \leq j \leq n$*

Remark 2.2 *If the fuzzy subset u is a discrete fuzzy number then the support of u coincides with its closure, i.e. $supp(u) = u^0$.*

From now on, the notation DFN stands for the set of discrete fuzzy numbers.

Remark 2.3 *In general, the operations on fuzzy numbers can be approached either by the direct use of their membership function as fuzzy subsets of $\mathbb{R}$ using the Zadeh's extension principle or by the equivalent use of the* α-cuts *representation[10]. Nevertheless, if u, v are discrete fuzzy numbers, these processes:*

$$(u \oplus v)(z) = \sup_{z=x+y} \min(u(x), v(y)), \forall z \in \mathbb{R}$$

$$MIN(u,v)(z) = \sup_{z=\min(x,y)} \min(u(x),v(y)), \forall z \in \mathbb{R}$$

$$MAX(u,v)(z) = \sup_{z=\max(x,y)} \min(u(x),v(y)), \forall z \in \mathbb{R}$$

can yield fuzzy subsets that do not satisfy the conditions to be discrete fuzzy numbers [3, 15].

In [3, 4, 5, 15], this drawback is studied and a new method to define these operations is proposed. So, let u, v be two discrete fuzzy numbers and $u^\alpha = \{x_1^\alpha, \cdots, x_p^\alpha\}$, $v^\alpha = \{y_1^\alpha, \cdots, y_k^\alpha\}$ their α-cuts respectively. And, for each pair of subsets $X, Y \subset \mathbb{R}$ and O a binary operation in $\mathbb{R}$, we will consider the set $XOY = \{z = xOy, x \in X, y \in Y\}$. Then, the next result holds [15]:

Theorem 2.4 *Let $u, v \in DFN$, the fuzzy subset denoted by $u \underset{W}{\oplus} v$, such that it has as r-cuts the sets $[u \underset{W}{\oplus} v]^r = \{x \in supp(u) + supp(v) : \min([u]^r + [v]^r) \leq x \leq \max([u]^r + [v]^r)\}$ for each $r \in [0,1]$ where $\min([u]^r + [v]^r) = \min\{x : x \in [u]^r + [v]^r\}$, $\max([u]^r + [v]^r) = \max\{x : x \in [u]^r + [v]^r\}$ and $(u \underset{W}{\oplus} v)(x) = \sup\{r \in [0,1]$ such that $x \in [u \underset{W}{\oplus} v]^r\}$ is a discrete fuzzy number.*

On the other hand, in [5], for each $u, v \in DFN$ the following sets are considered:

$$MIN_w(u,v)^\alpha = \{z \in supp(u) \bigwedge supp(v) \text{ such that}$$

$$\min(x_1^\alpha, y_1^\alpha) \leq z \leq \min(x_p^\alpha, y_k^\alpha)\} \text{ and}$$

$$MAX_w(u,v)^\alpha = \{z \in supp(u) \bigvee supp(v) \text{ such that}$$

$\max(x_1^\alpha, y_1^\alpha) \leq z \leq \max(x_p^\alpha, y_k^\alpha)\}$ for each $\alpha \in [0,1]$ where $supp(u) \bigwedge supp(v) = \{z = min(x,y) | x \in supp(u), y \in supp(v)\}$ and $supp(u) \bigvee supp(v) = \{z = max(x,y) | x \in supp(u), y \in supp(v)\}$.

And the following result is obtained:

Proposition 2.5 *[5] For each $u, v \in DFN$, there exist two unique discrete fuzzy numbers, which we will denote by $MIN_w(u,v)$ and $MAX_w(u,v)$, such that they have the above defined sets $MIN_w(u,v)^\alpha$ and $MAX_w(u,v)^\alpha$ as α-cuts respectively.*

3 Addition of Fuzzy Natural Numbers

Now, we wish to study some properties of the addition of fuzzy natural numbers (discrete fuzzy numbers whose support is a subset of consecutive natural numbers).

It is well known [10] that, in the case of continuous fuzzy numbers the addition obtained by extending the usual addition of real numbers through the extension principle is associative and commutative. But the fuzzy natural numbers are not continuous on $\mathbb{R}$.

In [4], the authors proved that in the case in which the discrete fuzzy numbers have as support an arithmetic sequence or a subset of consecutive natural numbers it is possible to use the Zadeh's extension principle to obtain its addition. Moreover, we know [15] the next result:

Proposition 3.1 *Let's consider $u, v \in DFN$. If $u \oplus v \in DFN$ where $u \oplus v$ denotes the addition of u and v using the Zadeh's extension principle, then $u \oplus v$ and $u \underset{W}{\oplus} v$ are identical, where $u \underset{W}{\oplus} v$ is the discrete fuzzy number obtained from u and v using theorem 2.4.*

Remark 3.2 *A consequence of the previous proposition is that if we prove a property for the operation $\underset{W}{\oplus}$ in the set of fuzzy natural numbers FNN, we will obtain the same property for the operation $\oplus$ in this set.*

From now on, the notation fnn stands for a fuzzy natural number.

Lemma 3.3 *Let u, v and w be three fnn such that their supports are $supp(u)$, $supp(v)$ and $supp(w)$ respectively. The following properties hold:*

1. *Commutativity*

$$supp(u) + supp(v) = supp(v) + supp(u)$$

2. *Associativity*

$$(supp(u) + supp(v)) + supp(w) =$$

$$= supp(u) + (supp(v) + supp(w))$$

Proof Let's denote by X, Y and Z the sets $supp(u)$,$supp(v)$ and $supp(w)$ respectively.

1. Using the commutative property of the addition of real numbers the proof is straightforward.

2. Associativity

   If $z \in (X+Y)+Z$ then $z = x+c$ where $x = a+b$, $a \in X$, $b \in Y$ and $c \in Z$. So $z = (a+b)+c$. Using the associativity of the addition of real numbers, $z = (a+b)+c = a+(b+c)$. Then $z \in X+(Y+Z)$. Therefore

$$(X+Y)+Z \subseteq X+(Y+Z)$$

   If $z \in X+(Y+Z)$ then $z = a+x$ where $x = b+c$, $a \in X$, $b \in Y$ and $c \in Z$. So $z = a+(b+c)$. Using the associativity of the addition of real numbers, $z = a+(b+c) = (a+b)+c$. Then $z \in (X+Y)+Z$. Therefore

$$X + (Y + Z) \subseteq (X + Y) + Z$$

Theorem 3.4 *For any, $u, v, w \in FNN$, the following properties hold:*

a) Commutativity:

$$u \underset{W}{\oplus} v = v \underset{W}{\oplus} u$$

b) Associativity:

$$(u \underset{W}{\oplus} v) \underset{W}{\oplus} w = u \underset{W}{\oplus} (v \underset{W}{\oplus} w)$$

c) Neutral element, i.e., $u \underset{W}{\oplus} \widehat{0} = u$ for all $u \in DFN$ where $\widehat{0}$ is the dfn $\widehat{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

Proof: Let u, v and w be three fnn. Let's consider the α-cut sets: $u^\alpha = \{x_1^\alpha, ..., x_p^\alpha\}$, $v^\alpha = \{y_1^\alpha, ..., y_k^\alpha\}$, $w^\alpha = \{w_1^\alpha, ..., w_l^\alpha\}$ for u, v and w respectively.

a) We want to show that

$$u \underset{W}{\oplus} v = v \underset{W}{\oplus} u$$

It is enough to prove that the fnn $u \underset{W}{\oplus} v$ and $v \underset{W}{\oplus} u$ are the same α-cut sets for each $\alpha \in [0,1]$, where $(u \underset{W}{\oplus} v)^0$ denotes the support of $u \underset{W}{\oplus} v$.

By definition $(u \underset{W}{\oplus} v)^\alpha = \{z \in supp(u) + supp(v)$ such that

$$\min\ (u^\alpha + v^\alpha) \leq z \leq\ \max\ (u^\alpha + v^\alpha)\} =$$

(by the monotonicity of the addition of natural numbers)$=\{z \in supp(u) + supp(v)$ such that

$$(\ \min u^\alpha +\ \min v^\alpha) \leq z \leq (\ \max u^\alpha +\ \max v^\alpha)\} =$$

$$\{z \in supp(u) + supp(v) | (x_1^\alpha + y_1^\alpha) \leq z \leq (x_p^\alpha + y_k^\alpha)\} =$$

$$\{z \in supp(u) + supp(v) | (y_1^\alpha + x_1^\alpha) \leq z \leq (y_p^\alpha + x_k^\alpha)\} =$$

$$(v \underset{W}{\oplus} u)^\alpha$$

b) We want to see that

$$(u \underset{W}{\oplus} v) \underset{W}{\oplus} w = u \underset{W}{\oplus} (v \underset{W}{\oplus} w)$$

By definition $((u \underset{W}{\oplus} v) \underset{W}{\oplus} w)^\alpha = \{z \in supp(u \underset{W}{\oplus} v) + supp(w)$ such that

$$\min((u \underset{W}{\oplus} v)^\alpha + w^\alpha) \leq z \leq \max((u \underset{W}{\oplus} v)^\alpha + w^\alpha)\} =$$

(by the monotonicity of the addition of natural numbers)$= \{z \in supp(u \underset{W}{\oplus} v) + supp(w)$ such that

$$\min(u \underset{W}{\oplus} v)^\alpha + \min w^\alpha \leq z \leq \max(u \underset{W}{\oplus} v)^\alpha + \max w^\alpha\} =$$

$\{z \in supp(u \underset{W}{\oplus} v) + supp(w)$ such that

$$(x_1^\alpha + y_1^\alpha) + w_1^\alpha \leq z \leq (x_p^\alpha + y_k^\alpha) + w_l^\alpha\} =$$

$\{z \in (supp(u) + supp(v)) + supp(w)$ such that

$$(x_1^\alpha + y_1^\alpha) + w_1^\alpha \leq z \leq (x_p^\alpha + y_k^\alpha) + w_l^\alpha\} =$$

$= \{z \in (supp(u) + supp(v)) + supp(w)$ such that

$$x_1^\alpha + (y_1^\alpha + w_1^\alpha) \leq z \leq x_p^\alpha + (y_k^\alpha + w_l^\alpha)\} =$$

$\{z \in supp(u) + (supp(v) + supp(w))$ such that

$$x_1^\alpha + (y_1^\alpha + w_1^\alpha) \leq z \leq x_p^\alpha + (y_k^\alpha + w_l^\alpha)\} =$$

$\{z \in supp(u) + (supp(v \underset{W}{\oplus} w))$ such that

$$x_1^\alpha + (y_1^\alpha + w_1^\alpha) \leq z \leq x_p^\alpha + (y_k^\alpha + w_l^\alpha)\} =$$

$$= (u \underset{W}{\oplus} (v \underset{W}{\oplus} w))^\alpha$$

c) Neutral element: $(u \underset{W}{\oplus} \widehat{0})(x) = (u \underset{W}{\oplus} \widehat{0})(x+0) = \sup_{z=x+0}(\min(u(x), \widehat{0}(0))) = \sup_{z=x+0}(\min(u(x), 1)) = u(x)$ for all $x \in supp(u)$.

Corollary 3.5 *The set FNN of the fuzzy natural numbers is a commutative monoid with the Zadeh's addition as a monoidal operation.*

4 Maximum and Minimum of Fuzzy Natural Numbers

With respect to the maximum and the minimum of two fuzzy natural numbers, the authors have proved in [5] the following proposition:

Proposition 4.1 *[5] Let u, v be two fuzzy natural numbers. Then $MAX(u,v)$, defined through the extension principle, coincides with $MAX_w(u,v)$. So, if $u, v \in FNN$, $MAX(u,v)$ is a fuzzy natural number and $MAX(u,v) \in FNN$. Analogously, $MIN(u,v)$, defined through the extension principle, coincides with $MIN_w(u,v)$. So, if $u, v \in FNN$, then $MIN(u,v)$ is a fuzzy natural number and $MIN(u,v) \in FNN$.*

But we have studied in [6] the associativity, commutativity, idempotence, absorption and distributivity for the operations MIN_w and MAX_w between discrete fuzzy numbers in general and between fuzzy natural numbers in particular and we obtained the following proposition:

Proposition 4.2 *[6] The set of discrete fuzzy numbers whose support is a sequence of consecutive natural numbers (FNN,MIN_w,MAX_w) is a distributive lattice.*

If we gather the previous propositions 4.1 and 4.2, then we obtain the following proposition:

Proposition 4.3 *[6] The set of discrete fuzzy numbers whose support is a sequence of consecutive natural numbers (FNN,MIN,MAX) is a distributive lattice.*

With the aim of studying the monotony for the addition of two fuzzy natural numbers, we need a definition of order:

Definition 4.4 *[6] Using the operations MIN_w and MAX_w, we can define a partial order on FNN on the following way:*
$u \preceq v$ if and only if $MIN_w(u, v) = u$, or equivalently, $u \preceq v$ if and only if $MAX_w(u, v) = v$ for any $u, v \in FNN$. Equivalently, we can also define the partial ordering in terms of α-cuts:

$u \preceq v$ *if and only if* $\min(u^\alpha, v^\alpha) = u^\alpha$

$u \preceq v$ *if and only if* $\max(u^\alpha, v^\alpha) = v^\alpha$

Proposition 4.5 *Let $u, v, w, t \in FNN$. If $u \preceq v$ and $w \preceq t$ where $\preceq$ denotes the partial order in FNN defined in definition 4.4 then $u \oplus w \preceq v \oplus t$, where $\oplus$ denotes the Zadeh's addition.*

Proof: From [6], we know that $u \preceq v$ iff $MIN(u, v) = u$ iff $\min(u^\alpha, v^\alpha) = u^\alpha$ for all $\alpha \in [0, 1]$. And analogously, $w \preceq t$ iff $MIN(w, t) = w$ iff $\min(w^\alpha, t^\alpha) = w^\alpha$ for all $\alpha \in [0, 1]$.

Let $u^\alpha = \{a_1^\alpha, \cdots, a_k^\alpha\}$, $v^\alpha = \{b_1^\alpha, \cdots, b_m^\alpha\}$, $w^\alpha = \{c_1^\alpha, \cdots, c_n^\alpha\}$ and $t^\alpha = \{d_1^\alpha, \cdots, d_p^\alpha\}$ the α-cuts of u, v, w and t respectively for $\alpha \in [0, 1]$. By definition of the addition of natural fuzzy numbers [4] the α-cuts of $u \oplus w$ and $v \oplus t$ are the sets of consecutive natural numbers $(u \oplus w)^\alpha = \{a_1^\alpha + c_1^\alpha, \cdots, a_k^\alpha + c_n^\alpha\}$ and $(v \oplus t)^\alpha = \{b_1^\alpha + d_1^\alpha, \cdots, b_m^\alpha + d_p^\alpha\}$.

Now, we want to prove that $MIN(u\oplus w, v\oplus t) = u\oplus w$, or equivalently [6], $MIN(u\oplus w, v\oplus t)^\alpha = (u\oplus w)^\alpha$ for all $\alpha \in [0, 1]$.

So, $MIN(u\oplus w, v\oplus t)^\alpha = \{z \in supp(u\oplus w) \wedge supp(u\oplus t) : \min((u\oplus w)^\alpha \wedge (v\oplus t)^\alpha) \leq z \leq \max((u\oplus w)^\alpha \wedge (v\oplus t)^\alpha)\}$=$\{z \in supp(u \oplus w) \wedge supp(v \oplus t) : a_1^\alpha + c_1^\alpha \leq z \leq a_k^\alpha + c_n^\alpha\}$=(as $u \oplus w$ has a set of consecutive natural numbers as a support and the addition is a monotone operation) $= \{z \in supp(u \oplus w) : a_1^\alpha + c_1^\alpha \leq z \leq a_k^\alpha + c_n^\alpha\}$= $(u \oplus w)^\alpha$.

5 Operations on FNN-VALUED MULTISETS

Definition 5.1 *A Fuzzy Natural Number-valued multiset defined over an universe X is a mapping $M : X \to FNN$ i.e. for all $x \in X$, $M(x)$ is a fuzzy natural number.*

Remark 5.2 *We will denote the set of Fuzzy Natural Number-valued multisets defined over an universe X by $FNNM(X)$. Finally, the abbreviation fnnm will denote a Fuzzy Natural Number-valued multiset.*

The properties of the addition of fuzzy natural numbers studied in the previous section, will allow us to define the addition of fuzzy natural number-valued multisets and to study the monoidal structure of this set.

5.1 Monoid structure

Definition 5.3 *Let $A, B : X \to FNN$ be two Fuzzy Natural Number-valued multisets. The* sum *of A and B will be the Fuzzy Natural Number-valued Multiset pointwise defined for all $x \in X$ by*

$$(A + B)(x) = A(x) \oplus B(x)$$

where the fnn $A(x) \oplus B(x)$ is obtained following the Zadeh's extension principle or equivalently using the method considered in theorem 2.4.

Proposition 5.4 *The set $FNNM(X)$ of the fuzzy natural number-valued multisets over X is a commutative monoid with the addition as a monoidal operation.*

Proof: The proof is straightforward from the corollary 3.5 and the definition of addition of fuzzy natural number-valued multisets 5.3.

5.2 Lattice structure and order

Analogously to the addition, the properties of the maximum and minimum of fnn studied in the previous section will allow us to define the maximum and minimum of fuzzy natural number-valued multisets and to study the order and the lattice structure of this set.

Definition 5.5 *Let $A, B : X \to FNN$ be two Fuzzy Natural Number-valued Multisets. The* join *and the* meet *of A and B will be the Fuzzy Natural Number-valued Multiset, pointwise defined for all $x \in X$ as*

$$(A \vee B)(x) = MAX\{A(x), B(x)\}$$

$$(A \wedge B)(x) = MIN\{A(x), B(x)\}$$

respectively,
where the fuzzy natural numbers $MAX\{A(x), B(x)\}$ and $MIN\{A(x), B(x)\}$ are obtained according to the method presented in proposition 2.5.

Proposition 5.6 *As long as, for all $x \in X$, $A(x) \in FNN$ and $B(x) \in FNN$, then $MAX\{A(x), B(x)\}$ and $MIN\{A(x), B(x)\}$ can be obtained by means of the extension principle.*

Proof: *By proposition 4.1, if $A(x) \in FNN$ and $B(x) \in FNN$, then the extension principle yields a discrete fuzzy number that it coincides with $MAX\{A(x), B(x)\} \in FNN$ (or $MIN\{A(x), B(x)\}$).*

Proposition 5.7 *Let $A, B : X \to FNN$ be two Fuzzy Natural Number-valued Multisets. The binary relationship:*
$A \leq B$ if and only if $A \vee B = B$ and/or $A \wedge B = A$ i.e. $MAX\{A(x), B(x)\} = B(x), \forall x \in X$ (or $MIN\{A(x), B(x)\} = A(x), \forall x \in X$) is a partial order on the set $FNNM(X)$.

Proposition 5.8 *The set $FNNM(X)$ of the fuzzy natural number-valued multisets over X is a lattice with the partial order defined in proposition 5.7 and the meet and join operations proposed in definition 5.5.*

Proof: The proof is straightforward from the lattice structure considered in proposition 4.2.

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