SPECIFICITY FOR INTERVAL-VALUED FUZZY SETS USING AGGREGATION OPERATORS

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Abstract

This paper proposes a new approach of measure of specificity for interval-valued fuzzy sets. It is showed a general expression of specificity which generalize the linear measure of specificity of Yager. Some examples are proposed.

Keywords: Specificity, Interval-valued fuzzy set.

1 Introduction

Interval-valued fuzzy sets ($\mathcal{IVFS}$) were introduced in the 60s by Grattan-Guinness [7], Jahn [8], Sambuc [9] and Zadeh [12]. They are extensions of classical fuzzy sets where the membership degree of the elements on the universe of discourse (between 0 and 1) is replaced by an interval in $[0,1] \times [0,1]$. They easily allow to model uncertainty and vagueness generalizing the fuzzy sets. Sometimes it is easier for experts to give a "membership interval" than a membership degree to objects on a universe. $\mathcal{IVFS}$ are a special case of type-2 fuzzy sets that simplifies the calculations while preserving their richness as well.

Fuzzy sets are a good tool to model imprecision due to vagueness. $\mathcal{IVFS}$ can also measure the imprecision due to uncertainty.

The concept of specificity provides a measure of the amount of information contained in a fuzzy set. It is strongly related to the inverse of the cardinality of a set. Specificity measures were introduced by Yager [10, 11] showing its usefulness as a measure of tranquility when making a decision. The output information of expert systems and other knowledgebased system should be both specific and correct to be useful.

Specificity for fuzzy sets has been widely analyzed in [3, 4, 5].

2 Preliminaries

Let $X = \{a_1, ..., a_n\}$ be a finite set.

Definition 2.1 *A fuzzy set μ on X is normal if there exists an element $x \in X$ such that $\mu(x) = 1$*

Definition 2.2 *[11] Let $[0,1]^X$ be the set of fuzzy sets on X. Let a_j be the j^{th} greatest membership degree of μ. A measure of specificity is a function Sp:$[0,1]^X \to [0,1]$ such that:*

- *$Sp(\mu) = 1$ if and only if μ is a singleton.*
- *$Sp(\emptyset) = 0$*
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  - $\frac{\partial Sp(\mu)}{\partial a_1} > 0$
  - $\frac{\partial Sp(\mu)}{\partial a_j} \leq 0$ *for all* $j \geq 2$

Definition 2.3 *[11] Let $[0,1]^X$ be the set of fuzzy sets on X. A weak measure of specificity is a function Sp:$[0,1]^X \to [0,1]$ such that:*

- *$Sp(\mu) = 1$ if and only if μ is a singleton.*
- *$Sp(\emptyset) = 0$*
- *If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $Sp(\mu) \geq Sp(\eta)$.*

Definition 2.4 *Let Sp and Sp' be two measures of specificity. Sp is more strict than Sp', denoted by $Sp \leq Sp'$, if for all set, μ, it verifies: $Sp(\mu) \leq Sp'(\mu)$.*

Yager indroduced [11] the linear measure of specificity on a finite space X as:

$$Sp_{\overrightarrow{w}}(\mu) = y_1 - \sum_{j=2}^{n} w_j y_j$$

where y_j is the j^{th} greatest membership degree of μ and $\{w_j\}$ is a set of weights verifying:

- $w_j \in [0,1]$
- $\Sigma_{j=2}^{n} w_j = 1$
- $\{w_j\}$ is not increasing.

Definition 2.5 *[2] Let $\mathcal{L}= (L, \leq_L)$ be a lattice that satisfies:*

1. $L = \{[x_1, x_2] \in [0,1]^2$ *with* $x_1 \leq x_2\}$.
2. $[x_1, x_2] \leq_L [y_1, y_2]$ *if and only if* $x_1 \leq y_1$ *and* $x_2 \leq y_2$

Also by definition:

$[x_1, x_2] <_L [y_1, y_2] \Leftrightarrow x_1 < y_1, x_2 \leq y_2$ or $x_1 \leq y_1, x_2 < y_2$
$[x_1, x_2] =_L [y_1, y_2] \Leftrightarrow x_1 = y_1, x_2 = y_2$.

$0_L =_L [0,0]$ and $1_L =_L [1,1]$ are the smallest and the greatest elements in L respectively.

$\mathcal{L}$ is a complete lattice and the supremum and infimum are defined as follows.

Definition 2.6 *[1] Let $\{[v_i, w_i]\}$ be a set of intervals on L. Then*

1. *Infimum:* $Meet\{[v_i, w_i]\} \equiv [infimun\{v_i\}, infimun\{w_i\}]$
2. *Supremum:* $Joint\{[v_i, w_i]\} \equiv [supremun\{v_i\}, supremun\{w_i\}]$

Definition 2.7 *[2] An interval-valued fuzzy set A on a universe X can be represented by the mapping:*

$$A = \{(a, [x_1, x_2]) \mid a \in X, [x_1, x_2] \in L\}$$

Definition 2.8 *[2] Let X be a universe and A and B two interval-valued fuzzy sets. The equality between A and B is defined as: $A =_L B$ if and only if $A(a) =_L B(a)$ $\forall a \in X$.*

Definition 2.9 *[2] Let X be a universe and A and B two interval-valued fuzzy sets. The inclusion of A in to B is defined as: $A \subseteq_L B$ if and only if $A(a) \leq_L B(a)$ $\forall a \in X$.*

Definition 2.10 *[2] A negation function for interval-valued fuzzy sets $\mathcal{N}$ is a decreasing function, $N : L \rightarrow L$, that satisfies:*

1. $N(0_L) =_L 1_L$
2. $N(1_L) =_L 0_L$

If $N(N([x_1, x_2])) =_L [x_1, x_2]$ then N is called an involutive negation.

Definition 2.11 *A strong negation function for interval-valued fuzzy sets, N, is a strictly decreasing and involutive function, $N : L \rightarrow L$, that satisfies:*

1. $N(0_L) =_L 1_L$
2. $N(1_L) =_L 0_L$

Example 2.1 *Let N be the involutive mapping defined by :*

$$N : L \rightarrow L$$
$$N([x_1, x_2]) =_L [1 - x_2, 1 - x_1]$$

Then N is a negation operator for interval-valued fuzzy sets. It is trivial to prove that: $N(0_L) =_L 1_L$, $N(1_L) =_L 0_L$ and $N(N([x_1, x_2])) =_L [x_1, x_2]$.

3 Specificity for Interval-valued Fuzzy Sets

Definition 3.1 *An operator $f : [0,1]^2 \rightarrow [0,1]$ with $x \leq y$ is called transformation operator if it is continuous, increasing and verifies:*

1. $f(1,1) = 1$
2. $f(0,0) = 0$
3. $f(0,x) > 0$ *for all* $x \in (0,1]$
4. $f(x,1) < 1$ *for all* $x \in [0,1)$

Definition 3.2 *Let μ be an interval-valued fuzzy set on X and let $\mu(a_q) = [x_{1_q}, x_{2_q}]$ be its membership intervals. Let f be a transformation operator. Then, the f-list of μ is the set of all the membership intervals of elements of X, ordered through the operator f, that is, $[x,y] \leq_f [z,t]$ if and only if $f(x,y) \leq f(z,t)$:*

Definition 3.3 *An interval-valued fuzzy set μ on X is a singleton if there exists an element $a_i \in X$ such that $\mu(a_i) = 1_L$ and $\mu(a_j) = 0_L$ (for all $j \neq i$) for the others.*

Definition 3.4 *Let $([0,1]^2)^X$ be the set of interval-valued fuzzy sets on X. Let f be a transformation operator. Let $\{[x_{1_q}, x_{2_q}]\}$ for all $q = 1..n$ be the f-list of μ. A f-measure of specificity for interval-valued fuzzy sets is a function $Sp_f : ([0,1]^2)^X \rightarrow [0,1]$ such that:*

- $Sp_f(\mu) = 1$ *if and only if* μ *is a singleton.*
- $Sp_f(\emptyset) = 0$.
- *If* $[x_{1_1}, x_{2_1}]$ *increases (according to* $\leq_L$*) then* $Sp_f(\mu)$ *increases.*
- *If* $[x_{1_q}, x_{2_q}]$ *increases (according to* $\leq_L$*) then* $Sp_f(\mu)$ *decreases for all* $q : 2..n$.

Definition 3.5 *An interval-valued fuzzy set* μ *on* X *is normal if there exists an element* $a \in X$ *such that* $\mu(a) = 1_L$

Definition 3.6 *[6] Let* $([0,1]^2)^X$ *be the set of interval-valued fuzzy sets on X. A weak measure of specificity for interval-valued fuzzy sets is a function* $Sp{:}([0,1]^2)^X \rightarrow [0,1]$ *such that:*

- $Sp(\mu) = 1$ *if and only if* μ *is a singleton.*
- $Sp(\emptyset) = 0$
- *If* μ *and* η *are normal fuzzy sets in* X *and* $\mu \subseteq_L \eta$, *then* $Sp(\mu) \geq Sp(\eta)$.

Lemma 3.1 *If* Sp_f *is a f-measure of specificity for interval-valued fuzzy sets then* Sp_f *is a weak measure of specificity for interval-valued fuzzy sets.*

Proof
Let $\{[x_{1_q}, x_{2_q}]\}$ and $\{[y_{1_q}, y_{2_q}]\}$ for all $q = 1..n$ be the f-list of μ and η respectively. If μ and η are normal and $\mu \subseteq_L \eta$ then $[x_{1_q}, x_{2_q}] \leq_L [y_{1_q}, y_{2_q}]$ for all $q = 2..n$. According to the fourth axiom of the definition 3.4 $Sp_f(\mu) \geq Sp_f(\eta)$.

Theorem 3.1 *Let* f *be a transformation operator and* $\{\alpha_i\}$ *a set of weights that satisfies:*

- $\alpha_j \in [0,1]$
- $\Sigma_{j=2}^{n} \alpha_j = 1$
- $\{\alpha_j\}$ *is not increasing.*

Let T, T', S *and* N *be, two t-norms, a t-conorm and a negation (in [0,1],≤) respectively. Let* $[x_{1_q}, x_{2_q}]$ *be the f-list of an interval-valued fuzzy set* μ. *Then, the next expression is a f-measure of specificity for interval-valued fuzzy sets:*

$$Sp_f(\mu) = T(f(x_{1_1}, x_{2_1}), N(S(T'(\alpha_2, f(x_{1_2}, x_{2_2}))), ..., T(\alpha_n, f(x_{1_n}, x_{2_n})))))$$

This expression is a generalization of the t-norm based measure of specificity given in [3] but extended for $\mathcal{IVFS}$.

Proof

1. $Sp_f(\mu) = 1$ if and only if μ is a singleton:
   - If μ is a singleton then $[x_{1_1}, x_{2_1}] = [1,1]$ and $[x_{1_k}, x_{2_k}] = [0,0]$ for all $k > 1$. Then $f(x_1, x_2)_1 = 1$ and $f(x_1, x_2)_k = 0$ for all $k > 1$.
   - If $Sp_f(\mu) = 1$, it is necessary that $f(x_1, x_2)_1 = 1$ and
   $$S(T(\alpha_2, f(x_1, x_2)_2)), ..., T(\alpha_n, f(x_1, x_2)_n) = 0$$
   Then $T(\alpha_k, f(x_1, x_2)_k)) = 0$ for all k and $f(x_1, x_2)_k = 0$ for all k.
2. $Sp_f(\emptyset) = 0$: trivial.
3. Trivial due to the fact T, T' and S are monotonic.

Example 3.1 *With* $T(a,b) = Max\{0, a+b-1\}$,
$N(a) = 1 - a$,
$S(a,b) = Min\{1, a+b\}$,
$T'(a,b) = a * b$ *and* $f(x,y) = \frac{x+y}{2}$, *it is obtained:*

$$Sp(\mu) = \frac{1}{2}(x_{1_1} + x_{2_1}) - \sum_{j=2}^{n} \alpha_j (x_{1_j} + x_{2_j})$$

Example 3.2 *With* $T(a,b) = Max\{0, a+b-1\}$,
$N(a) = 1 - a$,
$S(a,b) = Min\{1, a+b\}$,
$T'(a,b) = a*b$ *and* $f(x,y) = \alpha*x+\beta*y$ *with* $\alpha+\beta = 1$, *it is obtained:*

$$Sp(\mu) = \alpha * x_{1_1} + \beta * x_{2_1} - \sum_{j=2}^{n} \alpha_j (\alpha * x_{1_j} + \beta * x_{2_j})$$

Example 3.3 *With* $T(a,b) = Max\{0, a+b-1\}$,
$N(a) = 1 - a$,
$S(a,b) = Min\{1, a+b\}$,
$T'(a,b) = a * b$ *and* $f(x,y) = x * y$, *it is obtained:*

$$Sp(\mu) = x_{1_1} * x_{2_1} - \sum_{k>1} \alpha_k * x_{1_k} * x_{2_k}$$

Those examples 3.1, 3.2 and 3.3 are extensions of R. Yager's linear measure of specificity [11] for $\mathcal{IVFS}$.

4 Conclusions and future work

An approach of measures of specificity of interval-Valued fuzzy sets is given. it is based on the R. Yagers concept of measure of specificity as a measure of information that evaluates a degree of usefullness of information contained on a fuzzy set in order to make a decision.

A general expression of measures of specificity of interval-Valued fuzzy sets is given using t-norms, t-conorms a negation and a new proposed transformation operator in its definition, and some examples of this expression are provided using different connectives and different transformation operators.

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