FUZZY DECISION MAKING WITH THE PROBABILITY, THE WEIGHTED AVERAGE AND THE OWA OPERATOR

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Abstract

We develop a new decision making model based on the use of a unified approach between the probability, the weighted average and the ordered weighted averaging (OWA) operator in an uncertain environment that can be assessed with fuzzy numbers (FNs). We present a new aggregation operator called the fuzzy probabilistic ordered weighted averaging – weighted averaging (FPOWAWA) operator. We study its main properties and we find a wide range of particular cases such as the fuzzy probabilistic OWA, the fuzzy OWAWA, the fuzzy probabilistic weighted averaging operator and the fuzzy OWA operator. We also develop an application of the new approach in a decision making model about political management.

Keywords: Decision making, Probability, OWA operator, Weighted average, Fuzzy numbers.

1 INTRODUCTION

Decision making models are very useful in a lot of disciplines [4,6-12,15,17-18]. Some of the most common techniques for dealing with decision making are the probability and the weighted average (WA) [7]. Usually, they represent some kind of belief that the decision maker has about the problem, being the probability the objective perspective and the WA the subjective perspective. Another useful tool for decision making is the ordered weighted averaging (OWA) operator [15]. It is an aggregation operator that provides a parameterized family of aggregation operators between the maximum and the minimum. Thus, it is able to underestimate or overestimate the initial results of the decision maker according to a degree of optimism or pessimism. Since its appearance, the OWA operator has been studied by a lot of authors [1,4,6-18].

Several studies have tried to use the WA and the probability in the OWA operator. In [13], Torra developed a model for using OWAs and WAs in the same formulation called the weighted OWA (WOWA) operator. Later [14], Xu and Da developed another approach called the hybrid averaging (HA) operator. The used of the OWA operator in the probability has been analyzed in several papers. It is worth noting the work developed by [4,17] about immediate probabilities that has also been studied in [6-7,18]. Other studies that have used probabilistic information and OWAs are found in the analysis of the Dempster-Shafer theory of evidence [7,18]. More recent approaches for unifying the OWA with the probability and the WA are the probabilistic OWA (POWA) operator and the OWA weighted average (OWAWA) operator introduced in [7-9]. Another interesting approach is the one that unifies the probability with the weighted average, known as the probabilistic weighted averaging (PWA) operator [10]. A more general approach to the previous ones is the one suggested by Merigó in [11]. He suggested a unified model between the probability, the weighted average and the OWA operator. This new model is the probabilistic ordered weighted averaging – weighted averaging (POWAWA) operator.

Usually, when using these approaches it is considered that the available information are exact numbers. However, this may not be the real situation found in the specific problem considered. Sometimes, the available information is vague or imprecise and it is not possible to analyze it with exact numbers. Therefore, it is necessary to use another approach that is able to assess the uncertainty such as the use of fuzzy numbers (FNs). In order to develop the fuzzy approach, we will follow the ideas of [2-3,5,7,19-20].

The aim of this paper is to develop a new extension of the POWAWA operator for uncertain situations that can be represented with FNs. We will call it the fuzzy POWAWA (FPOWAWA) operator. The main advantage of this new approach is that it is able to unify the probability, the WA and the OWA in the same framework and considering the degree of importance that each concept has in the analysis. Moreover, this approach deals with uncertain environments where the available is not clear and it is necessary to use another approach such as the FNs. By using FNs, we are able to represent the information in a more complete way because we can consider the maximum and the minimum results and the possibility that the internal results of the fuzzy interval will occur.

We also study the applicability of the new approach and we see that it is very broad because it can be implemented in statistics, economics, engineering, etc. We focus on a decision making application about political management. We study a decision process where a national government is analyzing its optimal fiscal policy.

The paper is structured as follows. In Section 2, we briefly review some basic concepts. Section 3 presents the FPOWAWA operator. Section 4 develops a numerical example of the new approach and in Section 5 we summarize the main conclusions of the paper.

2 PRELIMINARIES

2.1. FUZZY NUMBERS

A FN A is defined as a fuzzy subset of a universe of discourse that is both convex (i.e., $\mu_A(\lambda x_1 + (1 - \lambda)x_2 \geq \min(\mu_A(x_1), \mu_A(x_2))$; for $\forall x_1, x_2 \in R$ and $\lambda \in [0, 1]$) and normal (i.e., $\sup_{x \in R} \mu_A(x) = 1$).

Note that the FN may be considered as a generalization of the interval number although it is not strictly the same because the interval numbers may have different meanings. In the literature, we find a wide range of FNs [2-3,5,7,19] such as the Triangular FN (TFN), the Trapezoidal FN (TpFN), the Interval-Valued FN (IVFN), the Generalized FN (GFN), and a lot of more complex structures.

For example, a TpFN A of a universe of discourse R can be characterized by a trapezoidal membership function (α–cut representation) $A = (\underline{a}, \overline{a})$ such that

$$\begin{aligned} \underline{a}(\alpha) &= a_1 + \alpha(a_2 - a_1), \\ \overline{a}(\alpha) &= a_4 - \alpha(a_4 - a_3). \end{aligned} \qquad (1)$$

where $\alpha \in [0, 1]$ and parameterized by (a_1, a_2, a_3, a_4) where $a_1 \leq a_2 \leq a_3 \leq a_4$, are real values. Note that if $a_1 = a_2 = a_3 = a_4$, then, the FN is a crisp value and if $a_2 = a_3$, the FN is represented by a TFN. Note that the TFN can be parameterized by (a_1, a_2, a_4).

In the following, we are going to review some basic FN arithmetic operations as follows. Let A and B be two TFNs, where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$.

1. $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$
3. $A \times k = (k \times a_1, k \times a_2, k \times a_3)$; for $k > 0$.

Note that other operations could be studied but in this paper we will focus on these ones. For more complete information and overviews about FNs, see for example [2-3,5,7,19].

2.2. THE OWA OPERATOR

The OWA operator [15] provides with a parameterized family of aggregation operators that include the arithmetic mean, the maximum and the minimum. It can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$OWA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \qquad (2)$$

where b_j is the jth largest of the a_i.

2.3. THE POWAWA OPERATOR

The probabilistic OWA weighted average (POWAWA) operator is an aggregation operator that uses probabilities, weighted averages and OWAs in the same formulation. It unifies these three concepts considering the degree of importance we want to give to each case depending on the situation considered. It can be defined as follows [7,11].

Definition 2. A POWAWA operator of dimension n is a mapping POWAWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{POWAWA}\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \qquad (3)$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, a probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$,

$\hat{v}_j = C_1 w_j + C_2 v_j + C_3 p_j$, with C_1, C_2 and $C_3 \in [0, 1]$, $C_1 + C_2 + C_3 = 1$, and v_j and p_j are the weights v_i and p_i ordered according to b_j, that is to say, according to the jth largest of the a_i.

3 THE FUZZY POWAWA OPERATOR

The FPOWAWA operator is an aggregation operator that unifies the probability, the WA and the OWA in the same formulation and considering the degree of importance that each concept has in the analysis. Thus, we can consider the objective and subjective information in the same problem and we are able to underestimate or overestimate the results according to our interests. Moreover, by using FNs we can represent the information considering the minimum and maximum results that may occur in the uncertain environment and the possibility (or membership level) that the internal values of the fuzzy interval will occur. The FPOWAWA operator can be defined as follows.

Definition 3. Let Ψ be the set of FNs. A FPOWAWA operator of dimension n is a mapping FPOWAWA: $\Psi^n \rightarrow \Psi$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{FPOWAWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \hat{v}_j b_j \quad (4)$$

where b_j is the jth largest of the $\tilde{a}_i$, each argument $\tilde{a}_i$ has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, a probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_j = C_1 w_j + C_2 v_j + C_3 p_j$, with C_1, C_2 and $C_3 \in [0, 1]$, $C_1 + C_2 + C_3 = 1$, and v_j and p_j are the weights v_i and p_i ordered according to b_j, that is to say, according to the jth largest of the $\tilde{a}_i$.

Note that this definition could also be presented using the following equivalent definition.

Definition 4. Let Ψ be the set of FNs. A FPOWAWA operator of dimension n is a mapping FPOWAWA: $\Psi^n \rightarrow \Psi$, that has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, a weighting vector V, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, and a probabilistic vector P, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, such that:

$$\text{FPOWAWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = C_1 \sum_{j=1}^{n} w_j b_j + C_2 \sum_{i=1}^{n} v_i \tilde{a}_i + C_3 \sum_{i=1}^{n} p_i \tilde{a}_i \quad (5)$$

where b_j is the jth largest of the $\tilde{a}_i$, the $\tilde{a}_i$ are FNs and C_1, C_2 and $C_3 \in [0, 1]$ with $C_1 + C_2 + C_3 = 1$.

As we can see, the OWA, the WA and the probability are included in this formulation as special cases.

- If $C_1 = 1$, we get the usual FOWA operator.
- If $C_2 = 1$, we get the usual fuzzy WA (FWA).
- If $C_3 = 1$, we get the usual fuzzy probability.
- If $C_1 = 0$, we form the fuzzy probabilistic weighted average (FPWA).
- If $C_2 = 0$, we form the fuzzy probabilistic OWA (FPOWA) operator.
- If $C_3 = 0$, we form the FOWAWA operator.

Note that it is possible to distinguish between the descending FPOWAWA (DFPOWAWA) and the ascending FPOWAWA (AFPOWAWA) operator by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DFPOWAWA and w^*_{n-j+1} the jth weight of the AFPOWAWA operator.

Note that if the weighting vectors of the three concepts are not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, $V = \sum_{i=1}^{n} v_i \neq 1$, $P = \sum_{i=1}^{n} p_i \neq 1$, then, the FPOWAWA operator can be expressed as:

$$\text{FPOWAWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \frac{C_1}{W} \sum_{j=1}^{n} w_j b_j + \frac{C_2}{V} \sum_{i=1}^{n} v_i a_i + \frac{C_3}{P} \sum_{i=1}^{n} p_i a_i \quad (6)$$

The FPOWAWA is monotonic, bounded and idempotent. Note that it is interesting to mention that with the FPOWAWA operator, we get new boundary conditions based on the minimum and the maximum probabilistic weighted aggregation. Other properties and particular cases will be considered in future research.

Note that it is possible to develop a more general formulation of the FPOWAWA operator that uses other concepts apart from the probability, the WA and the OWA. We believe that the FPOWAWA operator is a very natural unification but other unifications in a more artificial way could be developed such as the use of this formulation with the minimization of regret approach, the analytic hierarchy process (AHP), the TOPSIS approach, the utility model and a lot of other situations [7]. For doing so, instead of using a convex combination with three concepts we could use the following definition for m concepts.

Definition 5. Let Ψ be the set of FNs. A unified aggregation operator of dimension m is a mapping FUA: $\Psi^n \rightarrow \Psi$, that has associated m concepts with a degree of importance of C_h such that:

$$\text{FUA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{h=1}^{m} C_h K_h \quad (7)$$

where C_h is the degree of importance that each concept has in the aggregation, with $C_h \in [0, 1]$ and $\sum_{h=1}^{m} C_h = 1$, the arguments $\tilde{a}_i$ are FNs and K_h is the h concept considered in the aggregation.

As we can see, with this formulation we can unify in the same aggregation all the concepts we want. For example, with the FPOWAWA operator we have $h = 3$ and $K = 1$ is the OWA, $K = 2$ is the WA and $K = 3$ is the probabilistic aggregation. Note that these unified aggregations will be studied in future research but as we can see they follow a similar methodology than the POWAWA although we expect to find some other properties and particular cases.

Focussing on the FPOWAWA, if B is a vector corresponding to the ordered arguments b_j, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector, then, the FPOWAWA operator can be represented as follows:

$$\text{FPOWAWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = W^T B \quad (8)$$

A further interesting issue of the FPOWAWA operator is to analyze different particular cases such as the following ones obtained by using different representations in the weighting vectors.

- The fuzzy arithmetic PWA aggregation (if $w_j = 1/n$, for all j).
- The fuzzy arithmetic POWA operator (if $v_i = 1/n$, for all i).
- The fuzzy arithmetic OWAWA operator (if $p_i = 1/n$, for all i).
- The fuzzy double arithmetic OWA operator (if $p_i = 1/n$, for all i, and $v_i = 1/n$, for all i).
- The fuzzy double arithmetic WA operator (if $p_i = 1/n$, for all i, and $w_j = 1/n$, for all j).
- The fuzzy double arithmetic probabilistic aggregation (if $v_i = 1/n$, for all i, and $w_j = 1/n$, for all j).
- The fuzzy average (if $v_i = 1/n$, for all i, $p_i = 1/n$, for all i, and $w_j = 1/n$, for all j).
- The fuzzy maximum probabilistic weighted aggregation ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$).
- The fuzzy minimum probabilistic weighted aggregation ($w_n = 1$ and $w_j = 0$, for all $j \neq n$).
- The fuzzy maximum arithmetic probabilistic aggregation ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, and $v_i = 1/n$, for all i).
- The fuzzy maximum arithmetic WA aggregation ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, and $p_i = 1/n$, for all i).
- The fuzzy minimum arithmetic probabilistic aggregation ($w_n = 1$ and $w_j = 0$, for all $j \neq n$, and $v_i = 1/n$, for all i).
- The fuzzy minimum arithmetic WA aggregation ($w_n = 1$ and $w_j = 0$, for all $j \neq n$, and $p_i = 1/n$, for all i).
- The FPOWAWA Hurwicz criteria ($w_1 = \alpha$, $w_n = 1 - \alpha$ and $w_j = 0$, for all $j \neq 1, n$).
- The step-FPOWAWA ($w_k = 1$ and $w_j = 0$, for all $j \neq k$).
- The olympic-FPOWAWA operator ($w_1 = w_n = 0$, and $w_j = 1/(n - 2)$ for all others).
- The general olympic-FPOWAWA operator ($w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$).
- The S-FPOWAWA ($w_1 = (1/n)(1 - (\alpha + \beta) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta)$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$).
- The centered-FPOWAWA (if the weighting vector W is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).
- Etc.

Note that the FPOWAWA operator also includes the POWAWA operator as a particular case. Therefore, all this particular cases are also included in this approach. Furthermore, the uncertain POWAWA (UPOWAWA) operator is also included in this approach when the FNs are reduced to the interval numbers. These two cases are proved with the following theorems.

Theorem 1. If the FNs are reduced to the usual exact numbers, then, the FPOWAWA operator becomes the POWAWA operator [7,11].

Proof. Assume a TpFN = (a_1, a_2, a_3, a_4). If $a_1 = a_2 = a_3 = a_4$, then $(a_1, a_2, a_3, a_4) = a$, thus, we get the POWAWA operator.

Remark 1. In a similar way, we could develop the same proof for all the other types of FNs available in the literature [2-3,5,7,19].

Theorem 2. If the FNs are reduced to the interval numbers, then, the FPOWAWA operator becomes the uncertain POWAWA (UPOWAWA) operator [7].

Proof. Assume a TpFN = (a_1, a_2, a_3, a_4). If we only consider the points (a_1, a_2, a_3, a_4), then, the FN becomes an interval number (a quadruplet). Therefore, the FPOWAWA operator becomes the UPOWAWA operator.

Remark 2. In a similar way, we could develop the same proof for all the other types of FNs.

Remark 3. Note that similar analysis could be developed for considering situations when the FNs are representing linguistic variables, probabilistic sets, expertons, etc.

Note also that other families of FPOWAWA operators may be used following a similar methodology as it has been developed for the OWA operator and its extensions [1,4,6-18]. Moreover, we could extend this analysis to other types of uncertain approaches such as the use of interval numbers, linguistic approaches and expertons [7].

4 ILLUSTRATIVE EXAMPLE

The FPOWAWA operator and more general unifications can be applied in an unlimited number of applications because al the previous studies that have used these concepts (probability, WA and OWA) can be revised an extended with this new approach. Moreover, this approach is very general because it can represent the information by using FNs and all its particular cases such as the usual exact numbers.

Summarizing some of the main fields where it is possible to apply the POWA operator, we can mention: Statistics, Mathematics, Economics, Decision Theory, Engineering, Physics, Soft Computing, etc.

In this paper, we focus on an application in decision making about selection of fiscal policies. We analyze a country that is looking for its optimal fiscal policy for the next year and they consider 5 alternatives.

- A_1 = Develop a strong expansive fiscal policy.
- A_2 = Develop an expansive fiscal policy.
- A_3 = Do not develop any change.
- A_4 = Develop a contractive fiscal policy.
- A_5 = Develop a strong contractive fiscal policy.

In order to evaluate these policies, the government has brought together a group of experts. This group considers that the key factor is the economic situation of the world economy for the next period. They consider 5 possible states of nature that could happen in the future: N_1 = Very bad economic situation, N_2 = Bad economic situation, N_3 = Regular economic situation, N_4 = Good economic situation, N_5 = Very good economic situation.

The results of the available policies, depending on the state of nature N_i and the alternative A_k that the decision maker chooses, are shown in Table 1.

In this example, the experts assume the following weighting vectors: W = (0.1, 0.2, 0.2, 0.2, 0.3); V = (0.3, 0.2, 0.2, 0.2, 0.1); P = (0.3, 0.3, 0.2, 0.1, 0.1). Note that they assume that the OWA operator has an importance of 20%, the WA of 40% and the probabilistic information of 40%. With this information, we can aggregate the expected results in order to make a decision. In Table 2, we present different results by using different types of FPOWAWA operators.

Table 1: Available investment alternatives

	N_1	N_2	N_3	N_4	N_5
A_1	(60,70,80)	(40,50,60)	(30,40,50)	(70,80,90)	(80,90,100)
A_2	(20,30,40)	(10,20,30)	(80,90,100)	(90,100,110)	(70,80,90)
A_3	(60,70,80)	(60,70,80)	(50,60,70)	(30,40,50)	(60,70,80)
A_4	(80,90,100)	(20,30,40)	(30,40,50)	(40,50,60)	(80,90,100)
A_5	(90,100,110)	(20,30,40)	(40,50,60)	(60,70,80)	(60,70,80)

Table 2: Aggregated results

	FProb.	FWA	FOWA	FPOWAWA
A_1	(51,61,71)	(54,64,74)	(51,61,71)	(52.2,62.2,72.2)
A_2	(41,51,61)	(49,59,69)	(46,56,66)	(45.2,55.2,65.2)
A_3	(55,65,75)	(52,62,72)	(49,59,69)	(52.6,62.6,72.6)
A_4	(48,58,68)	(50,60,70)	(44,54,64)	(48,58,68)
A_5	(53,63,73)	(57,67,77)	(47,57,67)	(53.4,63.4,73.4)

In this case, we get as optimal policy A_5 for the FWA and the FPOWAWA, A_3 for the probabilistic approach and A_1 for the FOWA.

5 CONCLUSIONS

We have introduced the FPOWAWA operator. It is a new aggregation operator that unifies the probability, the WA and the OWA in the same formulation and considering the degree of importance that each concept has in the aggregation. Moreover, we have used FNs which are very useful for assessing uncertain environments considering the minimum and maximum results and the possibility that the internal values of the fuzzy interval will occur. We have studied some of its main properties and particular cases and we have developed a more general unified aggregation operator by using more concepts in the aggregation. We have also developed an application of the new approach in political management. Moreover, we have seen that the applicability of the FPOWAWA and more general aggregations is very broad because most of the sciences are affected by these concepts.

In future research, we expect to develop further developments of this approach by adding more characteristics such as the use of quasi-arithmetic means, distance measures and other types of uncertain information. We will also consider more general formulations by using more general unifications and other applications such as in group decision making.

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