Two-level interacting boson models beyond the mean field

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Summary

- The model.
- What is the valid order of the mean field?
- Holstein-Primakoff expansion plus Bogoliubov transformation.
  - Ground, one- and two-phonon state properties.
  - Spherical and deformed case.
  - Energies and number of bosons.
- Numerical results.
- Conclusions.

The Hamiltonian and the wave function (I)

- The Hamiltonian.

$$H = x\, n_L - \frac{1-x}{N} Q^{\chi} \cdot Q^{\chi},$$

$$Q^{\chi}_{\mu} = (s^{\dagger}\tilde{L} + L^{\dagger}s)^{(L)}_{\mu} + \chi[L^{\dagger} \times \tilde{L}]^{(L)}_{\mu},$$

where $\tilde{L}_{\mu} = (-1)^{\mu} L_{-\mu}$.

The Hamiltonian and the wave function (II)

- The wave function.

$$|c\rangle = \frac{1}{\sqrt{N!}}(\Gamma_c^\dagger)^N|0\rangle,$$

where for $L=0$

$$\Gamma_c^\dagger = \frac{1}{\sqrt{1+\beta^2}}\left(s^\dagger + \beta t^\dagger\right)$$

and for $L=2$

$$\Gamma_c^\dagger = \frac{1}{\sqrt{1+\beta^2}}\left(s^\dagger + \beta\cos\gamma d_0^\dagger + \frac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)\right),$$

The valid order of the mean field

- Schematic mean field energy for the ground state.

$$E(N,\ \beta,\ \gamma,\ x,\ \chi) = NF^{(1)}(\beta,\ \gamma,\ x,\ \chi) + (N-1)F^{(2)}(\beta,\ \gamma,\ x,\ \chi),$$

- Only the highest order in N is meaningful!
- The energy reduces to:

$$E_{N^1}(N,\ \beta,\ \gamma,\ x,\ \chi) = N\left[F^{(1)}(\beta,\ \gamma,\ x,\ \chi) + F^{(2)}(\beta,\ \gamma,\ x,\ \chi)\right].$$

$$E(N,\ \beta,\ x,\ y) = N\frac{\beta^2}{(1+\beta^2)^2}\left[5x - 4 + x\beta^2 + \beta y(x-1)(4+\beta y)\right],$$

where $y = \chi\langle L, 0; L, 0|L, 0\rangle$. In the case $L = 2$, $y = -\chi\sqrt{\frac{2}{7}}$.

The Holstein-Primakoff (HP) expansion (I)

- The HP transformation provides quantum corrections to the mean field through a precise clasification in powers of $1/N$:
  - It is a Hermitian transformation.
  - It preserves the boson conmutation relations.
  - It provides a correct expansion in N.
- The transformation:

$$
\begin{aligned}
L_\mu^\dagger L_\nu &= b_\mu^\dagger b_\nu, \\
L_\mu^\dagger s &= N^{1/2} b_\mu^\dagger (1 - n_b/N)^{1/2} = (s^\dagger L_\mu)^\dagger, \\
s^\dagger s &= N - n_b,
\end{aligned}
$$

where $[b_\mu, b_\nu^\dagger] = \delta_{\mu,\nu}$.

The Holstein-Primakoff (HP) expansion (II)

- We next introduce the c-bosons through a shift transformation

$$b_\mu^\dagger = \sqrt{N}\lambda_\mu^* + c_\mu^\dagger,$$

where the λ_μ's are complex numbers which form a $(2L+1)$-dimensional vector. We shall only consider the case $\lambda_0 \neq 0$ without loss of generality.

The Holstein-Primakoff (HP) expansion (III)

$$
\begin{aligned}
H \; = \; & N^1\lambda_0^2\left\{5x-4-4(x-1)\lambda_0^2+(x-1)\chi\alpha_{0,0}^{(L)}\lambda_0\left[4(1-\lambda_0^2)^{1/2}+\chi\alpha_{0,0}^{(L)}\lambda_0\right]\right\}+ \\
& N^{1/2}\lambda_0\left(c_0^\dagger+c_0\right)\left\{5x-4-8\lambda_0^2(x-1)+2(x-1)\chi\alpha_{0,0}^{(L)}\lambda_0\left[\frac{-4\lambda_0^2+3}{(1-\lambda_0^2)^{1/2}}+\chi\alpha_{0,0}^{(L)}\lambda_0\right]\right\}+ \\
& N^0\Bigg\{\left[3x-2-6\lambda_0^2(x-1)\right]n_c+(x-1)\Big[(2L+1) \\
& -(2L+3)\lambda_0^2+(1-\lambda_0^2)\left(P_c^\dagger+P_c\right)-4\lambda_0^2\left(c_0^{\dagger\,2}+2c_0^\dagger c_0+c_0^2\right)\Big]+ \\
& 2\chi(x-1)\Bigg\{\lambda_0(1-\lambda_0^2)^{1/2}\left[\sum_{\mu=-L}^{+L}\alpha_{0,\mu}^{(L)}+2c_\mu^\dagger c_\mu\left[(-1)^\mu\alpha_{\mu,-\mu}^{(L)}+\alpha_{0,\mu}^{(L)}\right]+(-1)^\mu\alpha_{\mu,0}^{(L)}\left(c_\mu^\dagger c_{-\mu}^\dagger+c_\mu c_{-\mu}\right)\right] \\
& -\frac{\lambda_0^3\alpha_{0,0}^{(L)}}{2(1-\lambda_0^2)^{1/2}}\left[2+3\left(c_0^{\dagger\,2}+2c_0^\dagger c_0+c_0^2\right)+2n_c\right]-\frac{\lambda_0^5\alpha_{0,0}^{(L)}}{4(1-\lambda_0^2)^{3/2}}\left(1+c_0^{\dagger\,2}+2c_0^\dagger c_0+c_0^2\right)\Bigg\}+ \\
& +\chi^2(x-1)\lambda_0^2\left\{1+\sum_{\mu=-L}^{+L}2c_\mu^\dagger c_\mu\left[(-1)^\mu\alpha_{0,0}^{(L)}\alpha_{\mu,-\mu}^{(L)}+\alpha_{0,\mu}^{(L)\,2}\right]+(-1)^\mu\alpha_{0,\mu}^{(L)\,2}\left(c_\mu^\dagger c_{-\mu}^\dagger+c_\mu c_{-\mu}\right)\right\}\Bigg\}+O(1/\sqrt{N})
\end{aligned}
$$

where $\alpha_{\mu,\nu}^{(L)}=\langle L,\mu;L\nu|L,\mu+\nu\rangle$ and $P_c^\dagger=c^\dagger\cdot c^\dagger=(P_c)^\dagger$. $\lambda_0=\beta_0/\sqrt{1+\beta_0^2}$.

Bogoliubov transformation

- We introduce a new kind of bosons, ξ, for transforming the Hamiltonian into a diagonal form.

$$
\begin{aligned}
c^{\dagger}_{\mu} &= u_{\mu}\xi^{\dagger}_{\mu} + v_{\mu}\tilde{\xi}_{\mu} \\
\tilde{c}_{\mu} &= u_{\mu}\tilde{\xi}_{\mu} + v_{\mu}\xi^{\dagger}_{\mu}
\end{aligned}
$$

where the coefficients verify $u_{\mu}^2 - v_{\mu}^2 = 1$, with $u_{\mu} = u_{-\mu}$ and $v_{\mu} = v_{-\mu}$.

The spherical case

- The Hamiltonian:

$$H = \frac{2L+1}{2}\left[-x+\Xi(x)^{1/2}\right] + n_\xi\,\Xi(x)^{1/2} + O(1/N),$$

where $\Xi(x) = x(5x-4)$ and n_ξ is the number operator for ξ bosons.

- The L-boson number operator:

$$\begin{aligned}
\langle n_L\rangle &= (1-x)^2\frac{\partial}{\partial x}\left[\frac{\langle H\rangle}{1-x}\right],\\
\langle n_L\rangle_{gs} &= \frac{2L+1}{2}\left[\frac{3x-2}{\Xi(x)}-1\right]+O(1/N),\\
\langle n_L\rangle_{p\xi} &= \langle n_L\rangle_{gs}+p\left[\frac{3x-2}{\Xi(x)}\right]+O(1/N).
\end{aligned}$$

The deformed case $(L = 2)$

- The Hamiltonian:

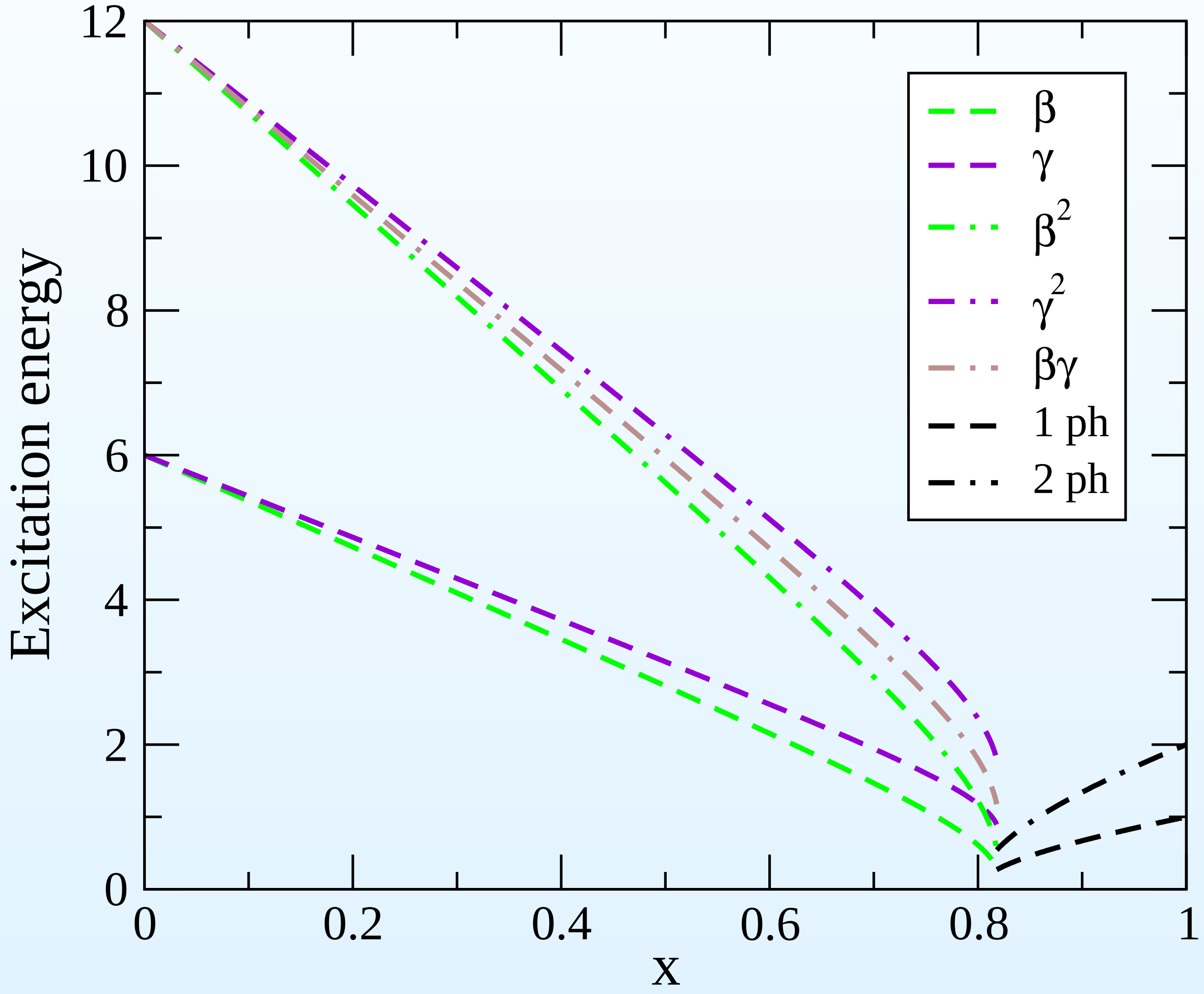
$$
\begin{aligned}
H &= E(x,y,\beta_0) + \frac{1}{2(1+\beta_0^2)}\left[-5x + (19x-24)\beta_0^2 + 12(x-1)y\beta_0^3\right] \\
&+ \sum_{\mu=-2}^{+2} \frac{\Phi_\mu(x,y,\beta_0)^{1/2}}{2} + n_{\xi_\mu}\Phi_\mu^{1/2}(x,y,\beta_0) + O(1/N),
\end{aligned}
$$

- The d-boson number operator:

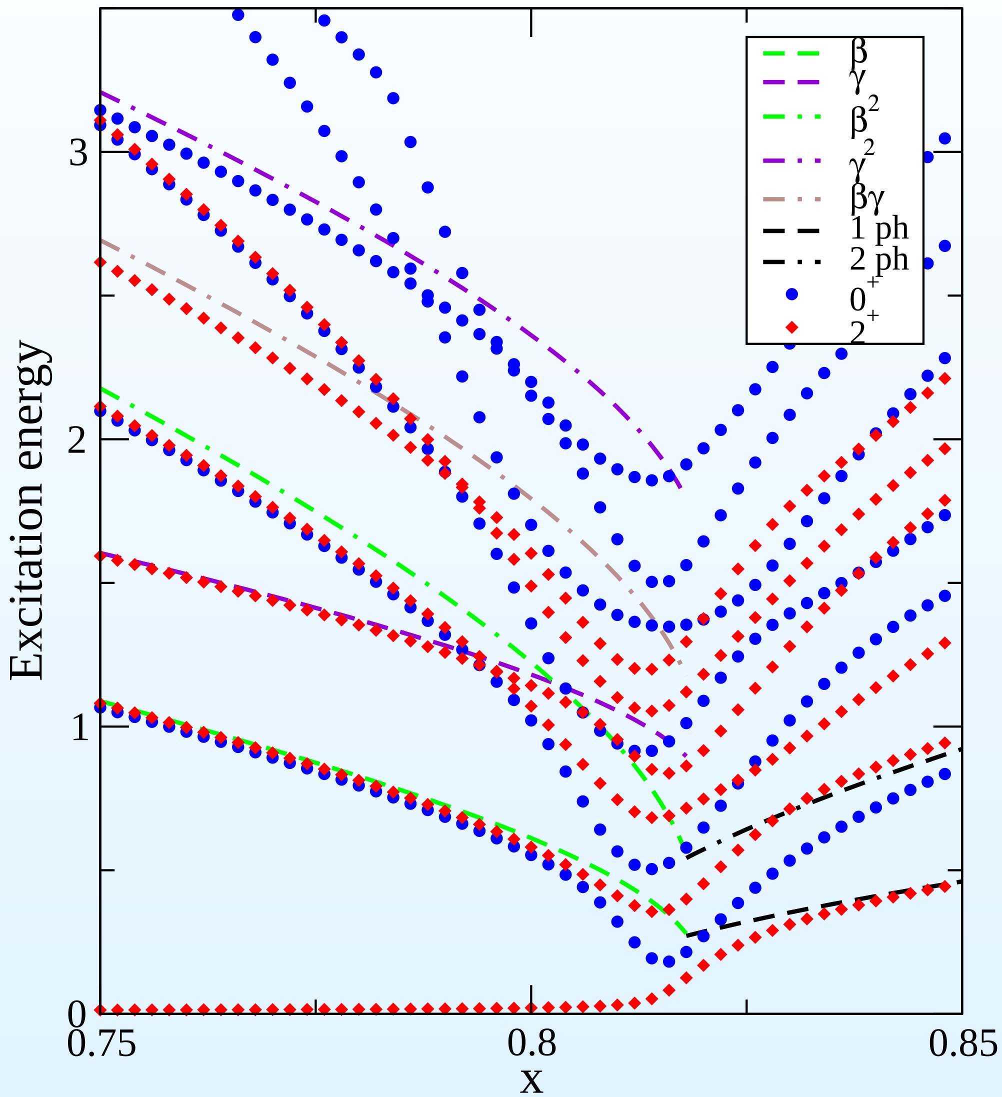
$$
\begin{aligned}
\langle n_L\rangle_{gs} &= N\frac{\beta_0^2}{1+\beta_0^2} + (1-x)^2\frac{\partial}{\partial x}\left(\frac{1}{2(1+\beta_0^2)(1-x)}\left[-5x + (19x-24)\beta_0^2\right.\right. \\
&+ \left.\left. 12(x-1)y\beta_0^3\right] + \sum_{\mu=-2}^{\mu=+2}\frac{\Phi_\mu(x,y,\beta_0)^{1/2}}{2(1-x)}\right) \\
\langle n_L\rangle_{p\xi_\mu} &= \langle n_L\rangle_{gs} + p(1-x)^2\frac{\partial}{\partial x}\left(\frac{\Phi_\mu(x,y,\beta_0)^{1/2}}{1-x}\right),
\end{aligned}
$$

- $\xi_0 \longrightarrow \beta\ , \quad \xi_{\pm 2} \longrightarrow \gamma.$

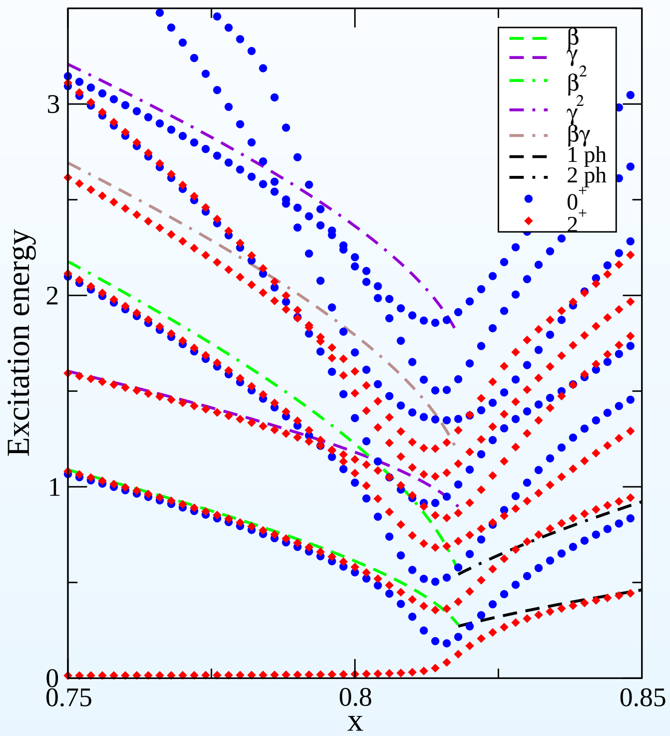
Excitation energies (analytical) for $L = 2$ and $\chi = -\sqrt{7}/2$



Exc. energies (analyt. and num.) for $L = 2$ and $\chi = -\sqrt{7}/2$, $N = 100$ in the critical area

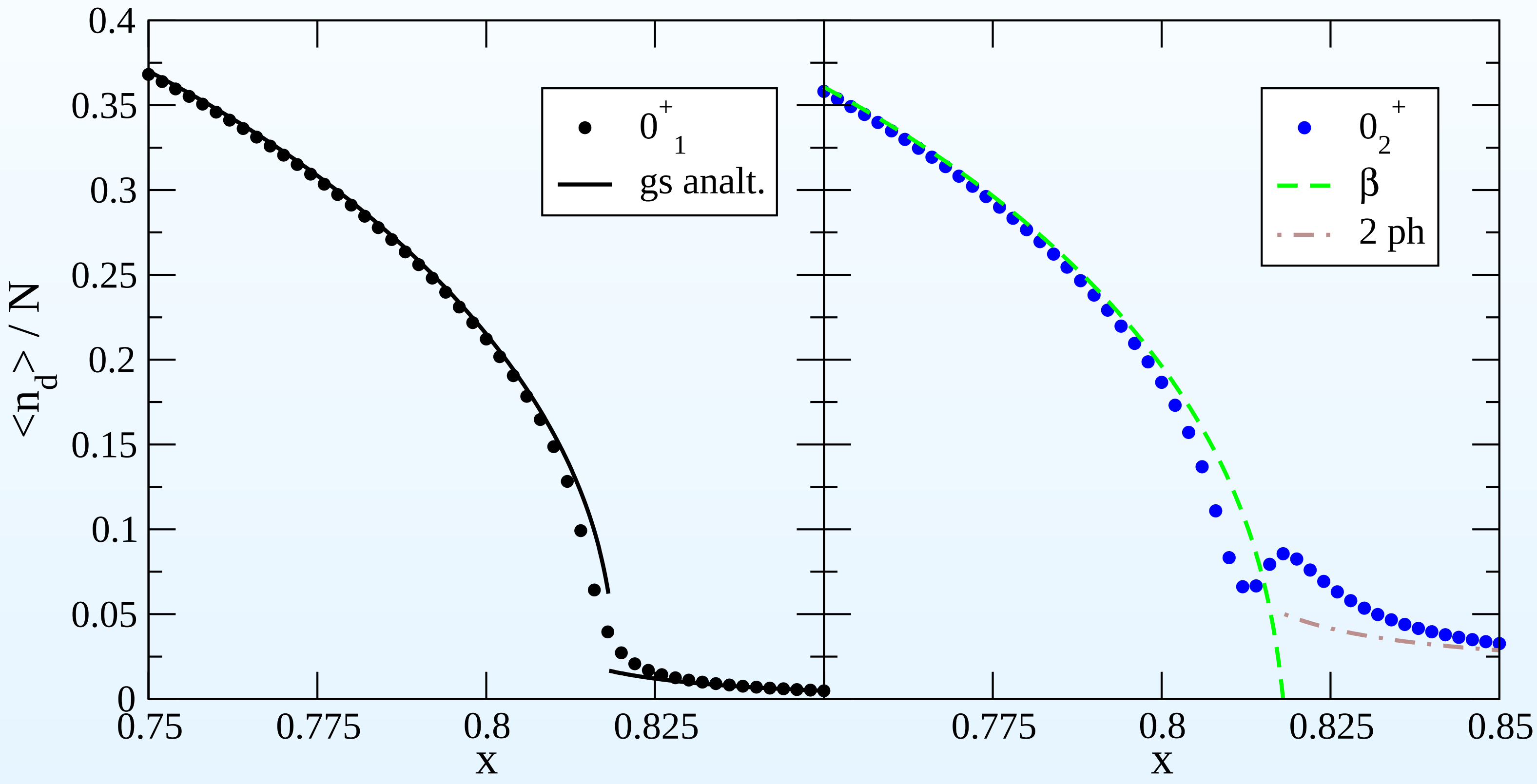


Exc. energies (analyt. and num.) for $L = 2$ and $\chi = -\sqrt{7}/2$, $N = 100$ in the critical area



	$x = 0.75$	$x = 9/11$	$x = 0.85$
β	$0_2^+, 2_2^+$	$0_2^+, 2_2^+$	
γ	2_3^+	2_4^+	
β^2	$0_3^+, 2_4^+$	$0_3^+, 2_3^+$	
$\beta\gamma$	2_5^+	2_6^+	
γ^2	0_5^+	0_7^+	
1 phonon			2_1^+
2 phonons			$0_2^+, 2_2^+$

$\langle n_d \rangle / N$ for $L = 2$ and $\chi = -\sqrt{7}/2$, $N = 100$



Summary and conclusions

- We have applied the HP+Bogoliubov expansion to two-level models for getting a $1/N$ expansion of the energy and other observables.
- The mean field energy is only valid for the higher order of N.
- We get a rather good description of the ground state, one and two phonon excitations around the critical area.

Some remarks on the critical area

- $\beta = 0$ is always a stationary point.
- For $\chi \neq 0$ there exists a region where two minima coexist. This region is delimited by the antispinodal $\beta = 0$ and the spinodal point:

$$\frac{3x}{3x-4} = \frac{\mathcal{A}}{\mathcal{B}}\left(1-\left(1+\frac{\mathcal{B}}{\mathcal{A}}\right)^{\frac{3}{2}}\right)$$

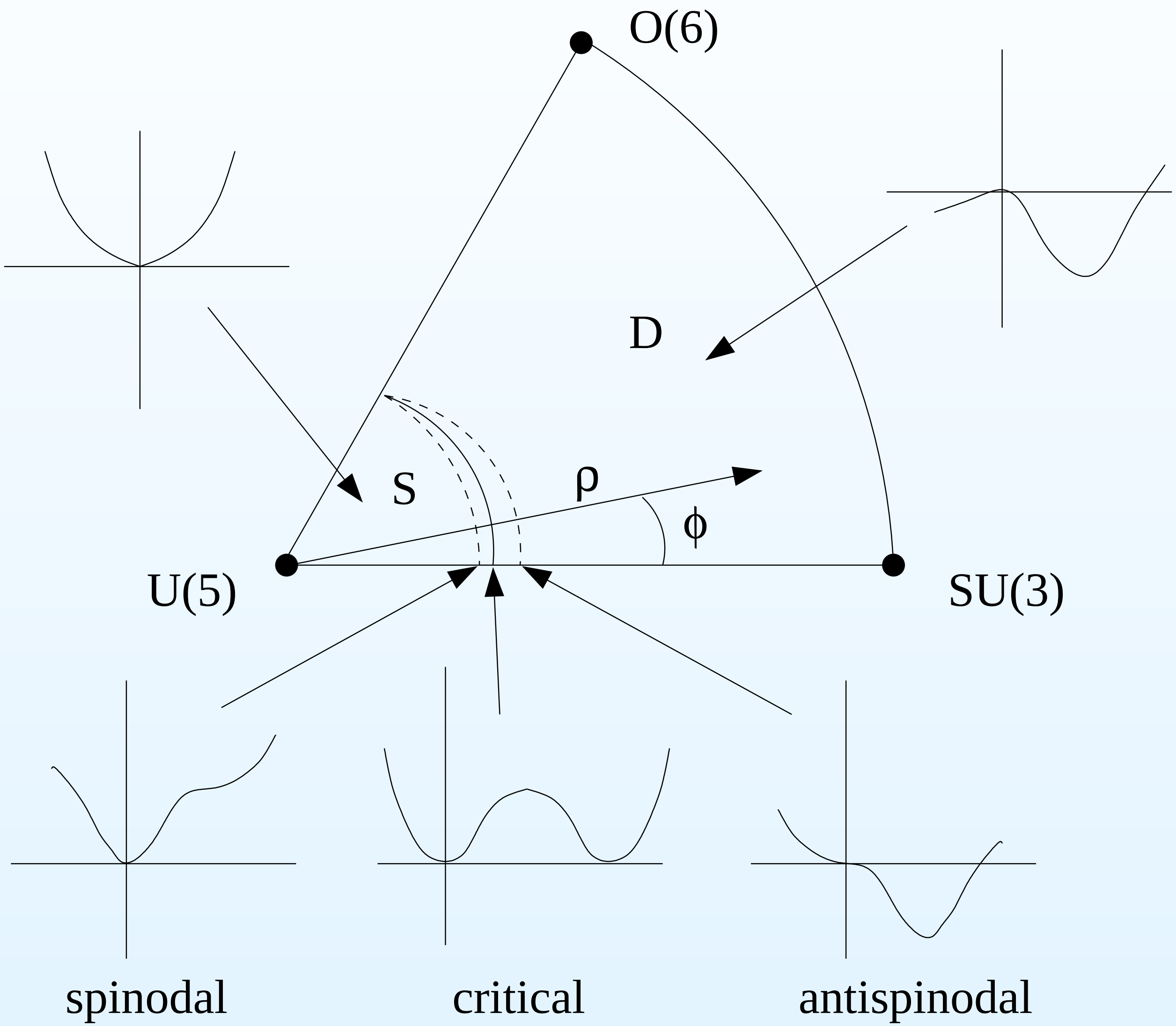
  where $\mathcal{A} = (4-3x+2(x-1)y^2)^2$ and $\mathcal{B} = 36y^2(x-1)^2$. In the $SU(3)$ case $x \approx 0.820361$.

- The critical point, x_c is defined as the situation in which the spherical and the deformed minima are degenerated.

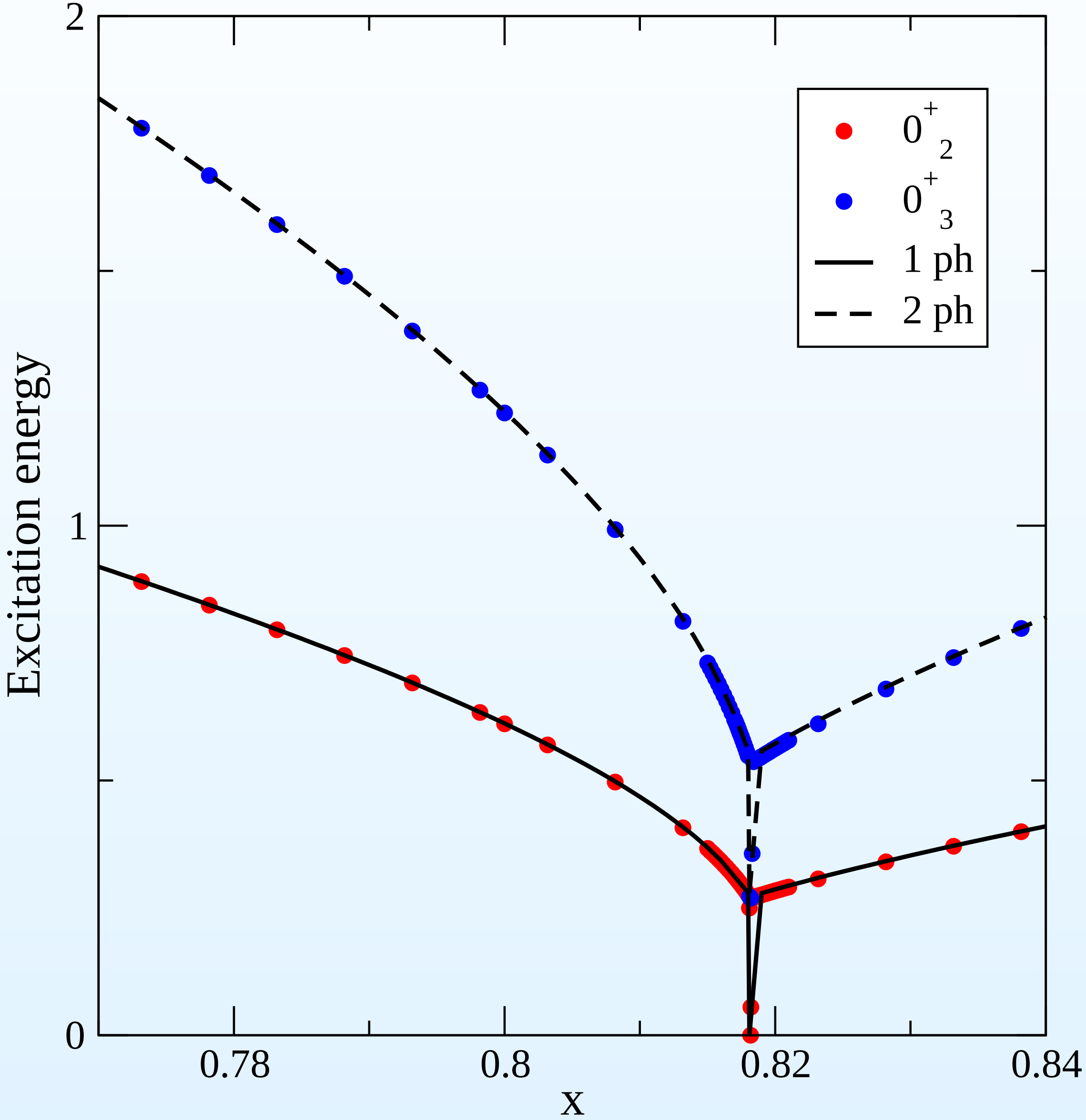
$$x_c = \frac{4+y^2}{5+y^2} = \frac{4+\chi^2\langle L,0;L0|L,0\rangle^2}{5+\chi^2\langle L,0;L0|L,0\rangle^2}.$$

  In the $SU(3)$ limit $x_c = 9/11$.

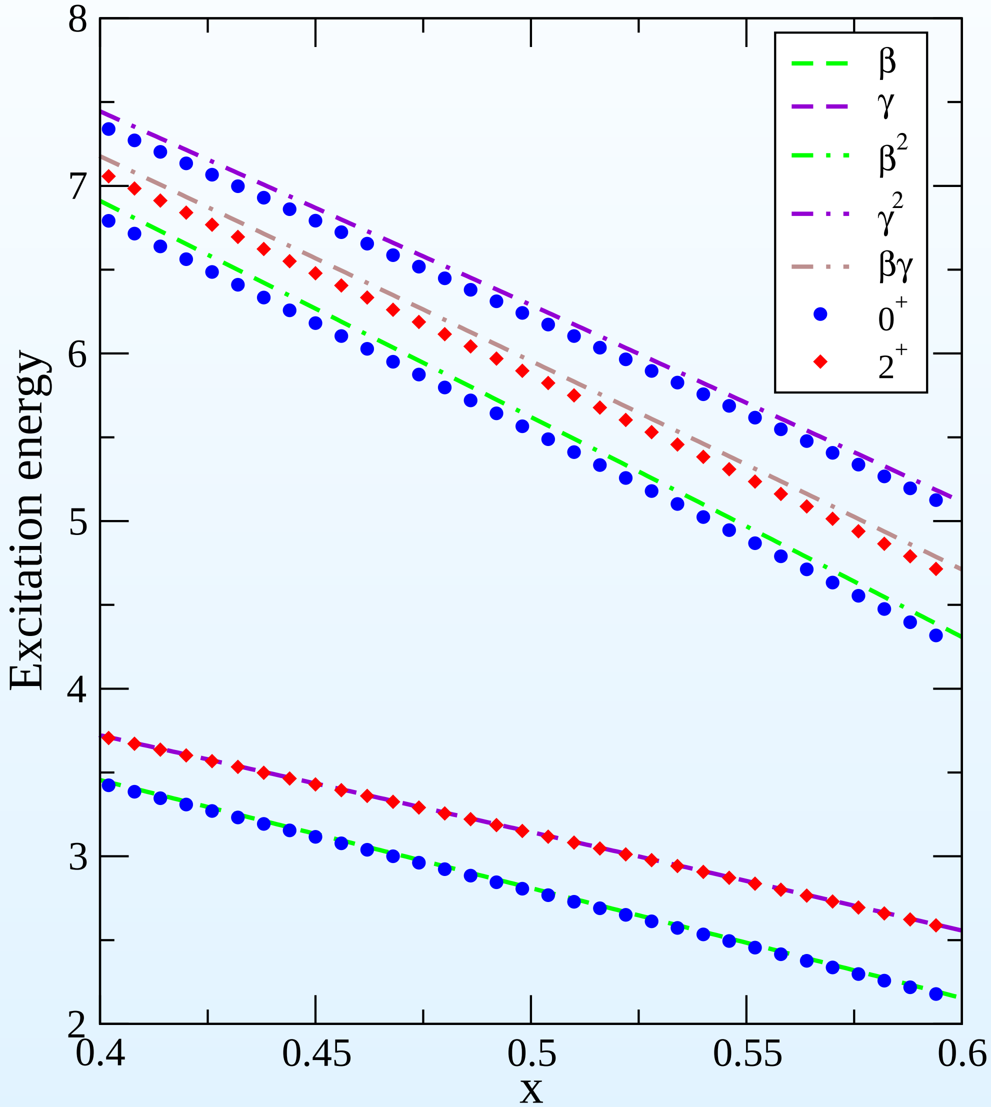
The phase diagram of IBM



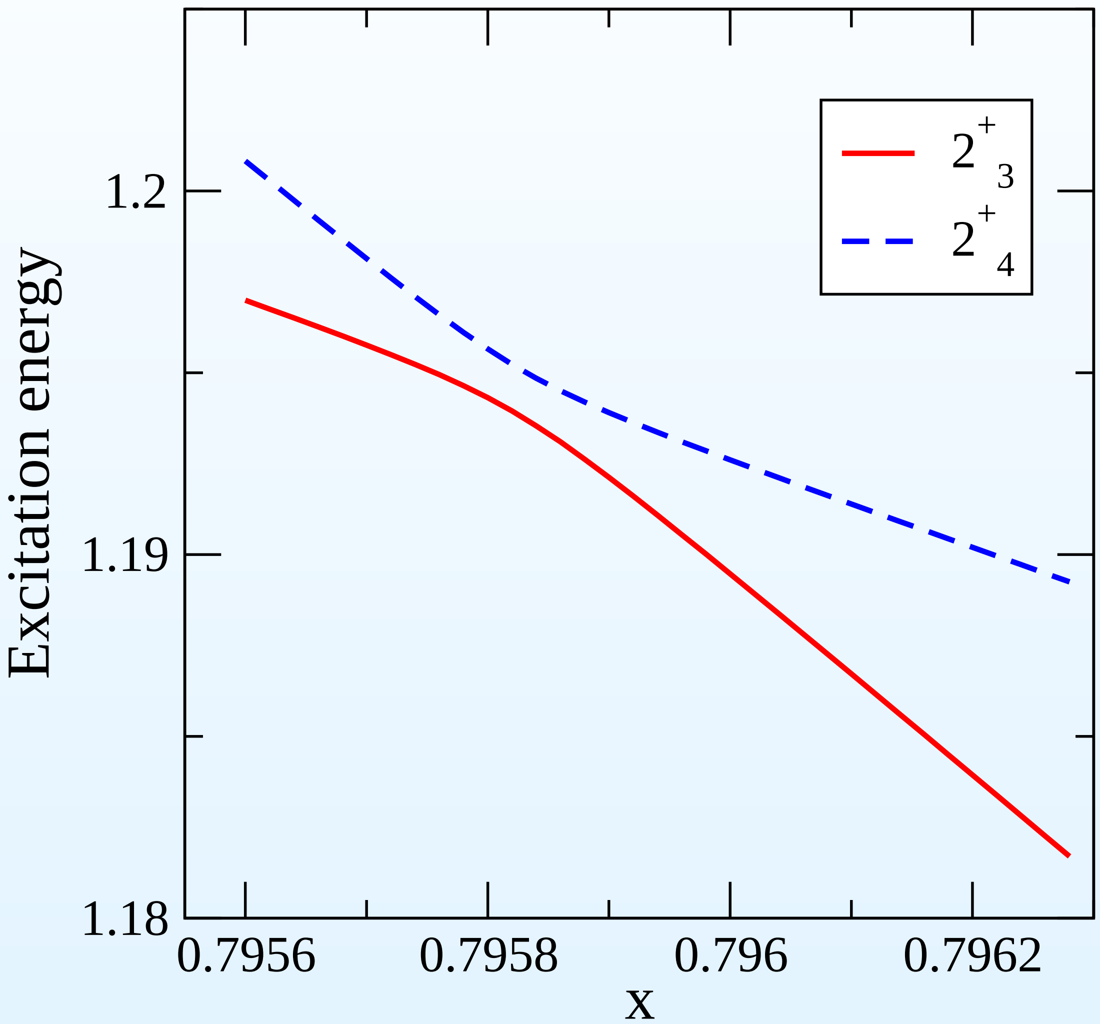
Excitation energies for $L = 0$, $N = 1000$ and $\chi = -\sqrt{7}/2$



Exc. energies (analyt. and num.) for $L = 2$ and $\chi = -\sqrt{7}/2$, $N = 100$ in the deformed ph.



Level repulsion $L = 2$ and $\chi = -\sqrt{7}/2$, $N = 100$: energies



Level repulsion $L = 2$ and $\chi = -\sqrt{7}/2$, $N = 100$: $\langle n_d \rangle / N$