Classical and quantum Lyapunov exponents as indicators of excited-state quantum phase transitions.

Sergio Lerma-Hernández
(Universidad Veracruzana, Mexico)

In collaboration with

Saúl Pilatowsky-Cameo
(Bachelor in Physics, UNAM)
Jorge Chávez-Carlos
(postdoc, Instituto Ciencias Física-UNAM)
Miguel Bastarrachea-Magani
(now in Universidad Autónoma Metropolitana-México)
Pavel Stránský
(Faculty of Mathematics and Physics, Charles University, Czech Republic)
Lea Santos
(Physics Department, Yeshiva University, NY, USA)
Jorge Hirsch
(Instituto de Ciencias Nucleres-UNAM)

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Classical and quantum Lyapunov exponents as indicators of excited-state quantum phase transitions

(a classical analysis).

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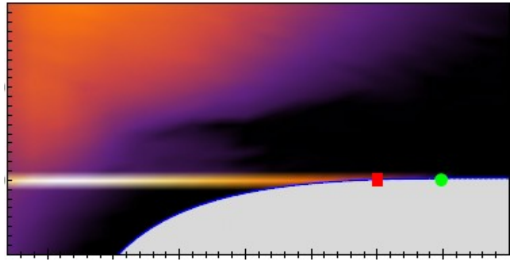
Outline

- Classical and quantum Lyapunov exponents, Out-of-Time-Order Correlators (OTOCs)
- Positive Lyapunov exponents are also associated to hyperbolic unstable fixed points in regular systems.
- A detector of Excited-States Quantum Phase Transitions: examples in the Lipkin-Meshkov-Glick model and Dicke model.
- Conclusions

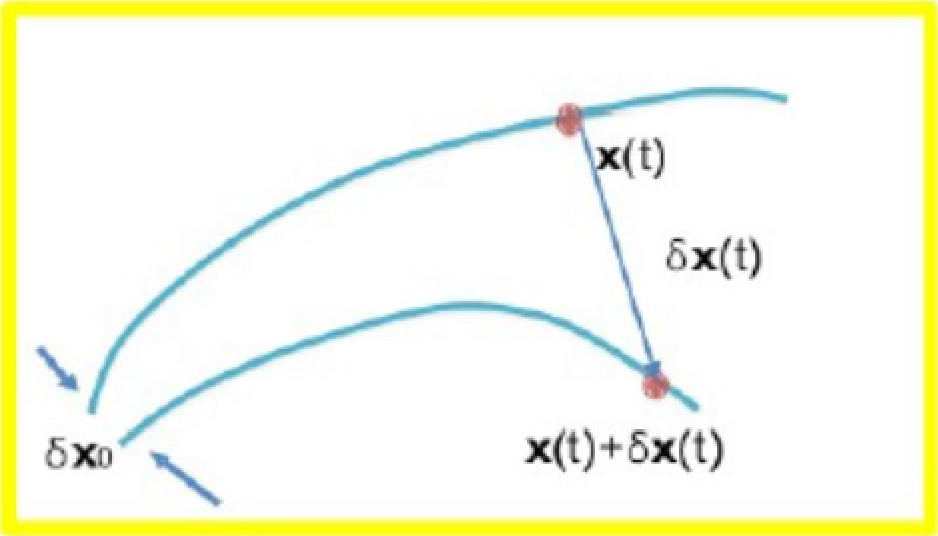


Sensitivity to initial conditions: classical Lyapunov exponent

Butterfly effect

$$\|\delta \mathbf{x}(t)\| \sim e^{\lambda t}\,\|\delta \mathbf{x}(0)\|$$

$$\lambda = \lim_{t\to\infty}\ \lim_{\|\delta \mathbf{x}_0\|\to 0}\ \frac{1}{t}\ln\frac{\|\delta \mathbf{x}(t)\|}{\|\delta \mathbf{x}_0\|}$$



Tangent space.

$$\boldsymbol{x} = (\boldsymbol{q},\boldsymbol{p}) = (q_1, \ldots, q_n, p_1, \ldots p_n)$$

Initial conditions

$$\boxed{\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\Phi}}\end{pmatrix} = \begin{pmatrix}F(\mathbf{x})\\ D_{\mathbf{x}}F(\mathbf{x})\boldsymbol{\Phi}\end{pmatrix}} \qquad \boxed{\begin{pmatrix}\mathbf{x}(\mathbf{t_0})\\ \boldsymbol{\Phi}(t_0)\end{pmatrix} = \begin{pmatrix}\mathbf{x_0}\\ \mathbb{I}\end{pmatrix}}$$

$$\delta \mathbf{x}(t) = \boldsymbol{\Phi}_t(\mathbf{x}_0)\cdot\delta \mathbf{x}_0$$

Out-of-Time order correlator (OTOC an information scrambling measure) :

Revival of an old concept: *Quasiclassical method in the theory of Superconductivity*
I. Larkin and Yu. N. Ovchinnikov Zh. Eksp. Teor. Fiz. 55, 2262-2272 (June, 1969)

JHEP08(2016)106

A bound on chaos

$$\boxed{\lambda_L \leq \frac{2\pi k_B T}{\hbar}}$$

Juan Maldacena,[a] Stephen H. Shenker[b] and Douglas Stanford[a]

In quantum mechanics, chaos can be characrterized using the commutator [W(t),V(0)] between rather general Hermitian operators at time separation t... One indication of the strenght of such effect is

$$C(t) = -\langle [W(t), V(0)]^2 \rangle$$

Para W(t)=q(t) y V(0)=p [q(t),p] $\longrightarrow$ $i\hbar\{q(t),p\} = i\hbar\frac{\partial q(t)}{\partial q(0)}$

$$C(t) \sim \hbar^2 e^{2\lambda_L t}$$

- MOTOC (Microcanonical OTOC)
- FOTOC (Fidelity OTOC)

Λ= quantum Lyapunov exponent

MOTOC

$$C_n^{qp}(t) = -\langle\Psi_n|[q(t),p(0)]^2|\Psi_n\rangle \sim e^{2\Lambda t}$$

$|\Psi_n\rangle$ Hamiltonian eigenstates

FOTOC

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger\hat{W}(t)\hat{V}\rangle$$

$$\hat{W} = e^{i\delta\phi\hat{G}}$$

$$\hat{V} = |\Psi_0\rangle\langle\Psi_0|$$

$$\frac{1-F(t)}{(\delta\phi)^2} \approx (\langle\hat{G}(t)^2\rangle - \langle\hat{G}(t)\rangle^2) \equiv \sigma_G^2(t) \sim e^{2\Lambda t}$$

Dicke Hamiltonian

$$H = \omega a^\dagger a + \omega_o J_z + \frac{\gamma}{\sqrt{N}}\left(a + a^\dagger\right)\left(J_+ + J_-\right)$$

Bosons energy — Two levels' energy splitting — Interaction term

$\hbar = 1$

$$J_z = \frac{1}{2}\sum_{i=1}^{N}\sigma_{iz} = \frac{1}{2}(N_e - N_g) \qquad J_\pm = \frac{1}{2}\sum_{i=1}^{N}\sigma_{i\pm}$$

$J=N/2$

One of the simplest autonomous Hamiltonian (time independent) with regular and **chaotic** regimes. Excellent playground to test ideas.

Experimentally accesible on different platforms

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Dicke-model simulation via cavity-assisted Raman transitions

Zhiqiang Zhang, Chern Hui Lee, Ravi Kumar, K. J. Arnold, Stuart J. Masson, A. L. Grimsmo, A. S. Parkins, and M. D. Barrett
Phys. Rev. A **97**, 043858 – Published 25 April 2018

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Ultrastrong-coupling phenomena beyond the Dicke model

Tuomas Jaako, Ze-Liang Xiang, Juan José Garcia-Ripoll, and Peter Rabl
Phys. Rev. A **94**, 033850 – Published 27 September 2016

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Verification of a Many-Ion Simulator of the Dicke Model Through Slow Quenches across a Phase Transition

A. Safavi-Naini, R. J. Lewis-Swan, J. G. Bohnet, M. Gärttner, K. A. Gilmore, J. E. Jordan, J. Cohn, J. K. Freericks, A. M. Rey, and J. J. Bollinger
Phys. Rev. Lett. **121**, 040503 – Published 27 July 2018

By using coherent states…

Coherent states

bosonic $|z\rangle = e^{|z|^2/2} e^{z a^\dagger} |0\rangle_b$

$z, \alpha \in \mathcal{C}$

pseudospin $|\alpha\rangle = \frac{1}{(1+|\alpha|^2)^j} e^{\alpha J_+} |jm = -j\rangle$

We can define a classical Hamiltonian:

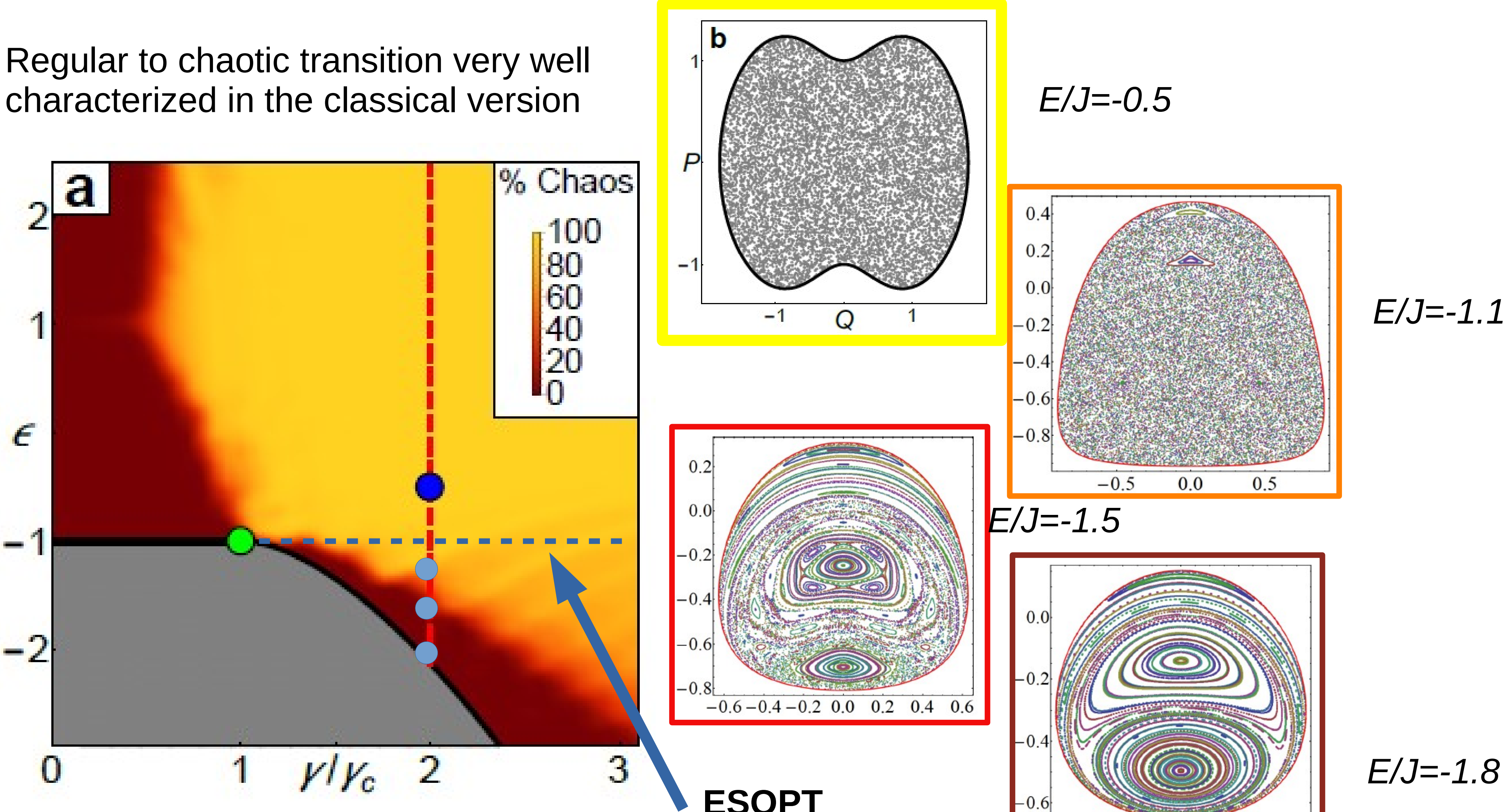
$$\langle \alpha, z | H | \alpha, z \rangle / j = \frac{\omega}{2}(q^2 + p^2) - \omega_0 + \frac{\omega_0}{2}(Q^2 + P^2) + 2\gamma \sqrt{1 - \frac{1}{4}(Q^2 + P^2)}\, qQ$$

$$z = \sqrt{j/2}(q + ip)$$

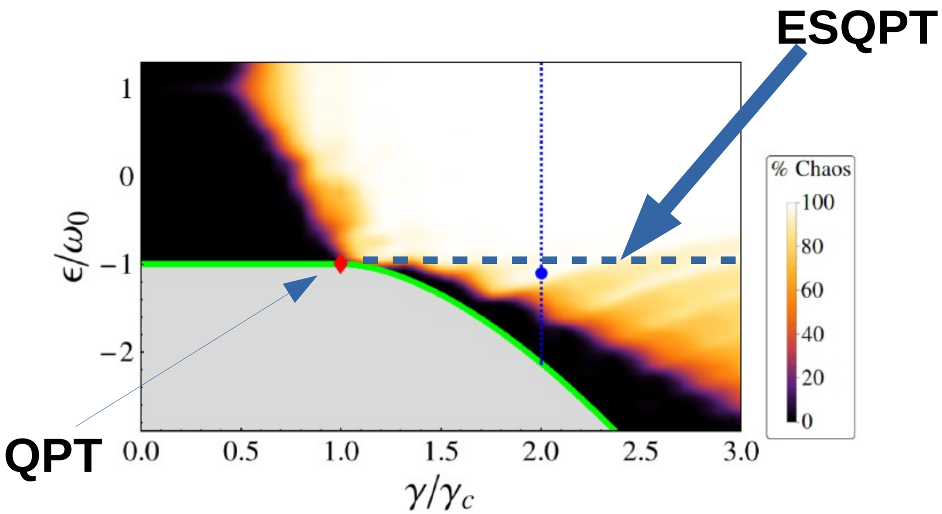
$$\alpha = (Q - iP)/\sqrt{4 - (Q^2 + P^2)}$$

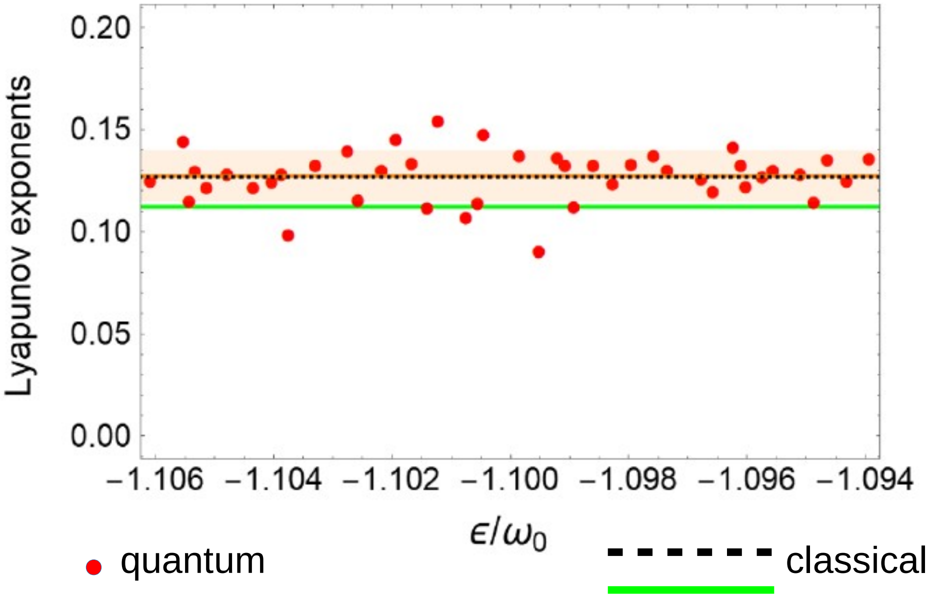
$\hbar_{eff} = 1/J$

Regular to chaotic transition very well characterized in the classical version



Physical Review E 94 (2), 022209 (2016)

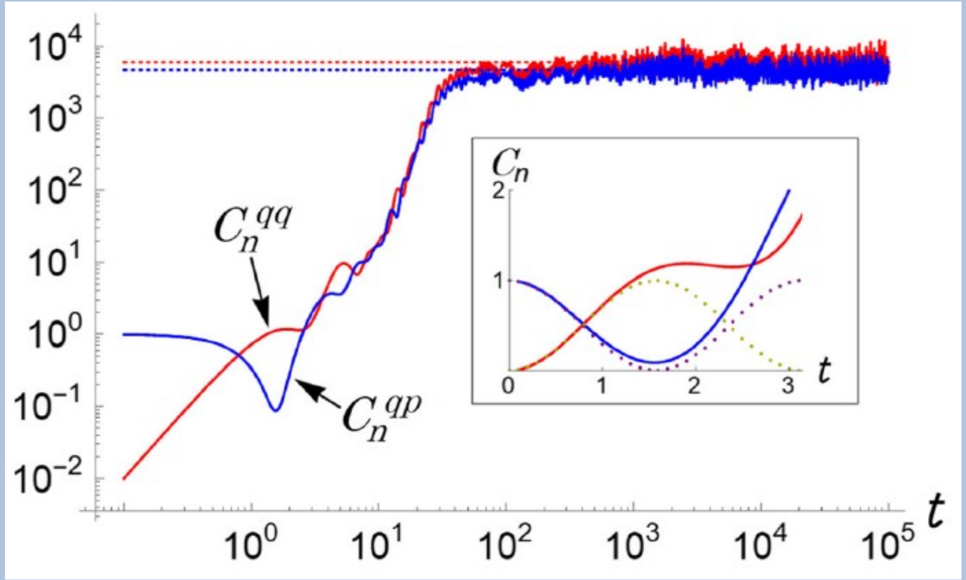




MOTOC in chaotic energy region

$$C_n^{qq}(t) = -\langle\Psi_n|[q(t),q(0)]^2|\Psi_n\rangle$$

$$C_n^{qp}(t) = -\langle\Psi_n|[q(t),p(0)]^2|\Psi_n\rangle$$



Article | Open Access | Published: 05 April 2019

Unifying scrambling, thermalization and entanglement through measurement of fidelity out-of-time-order correlators in the Dicke model

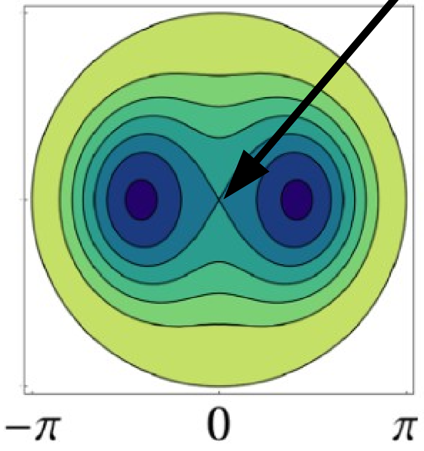
R. J. Lewis-Swan, A. Safavi-Naini, J. J. Bollinger & A. M. Rey

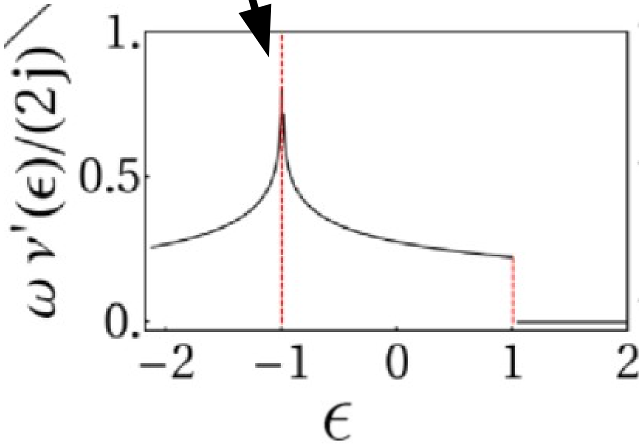
Nature Communications **10**, Article number: 1581 (2019) | Cite this article

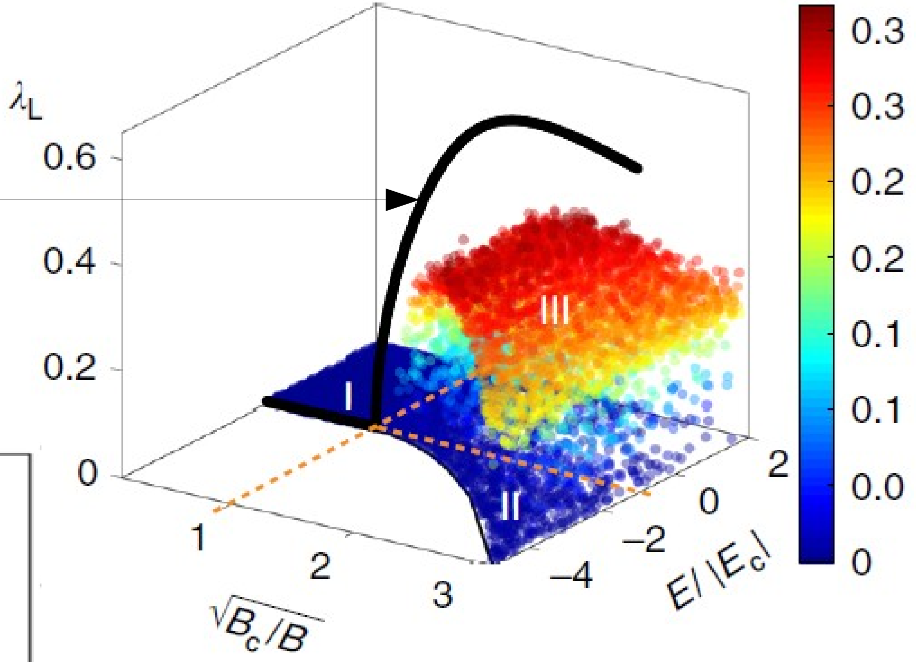
Lyapunov exponent

Large Lyapunov exponents at the ESQPT energy

Classical phase space



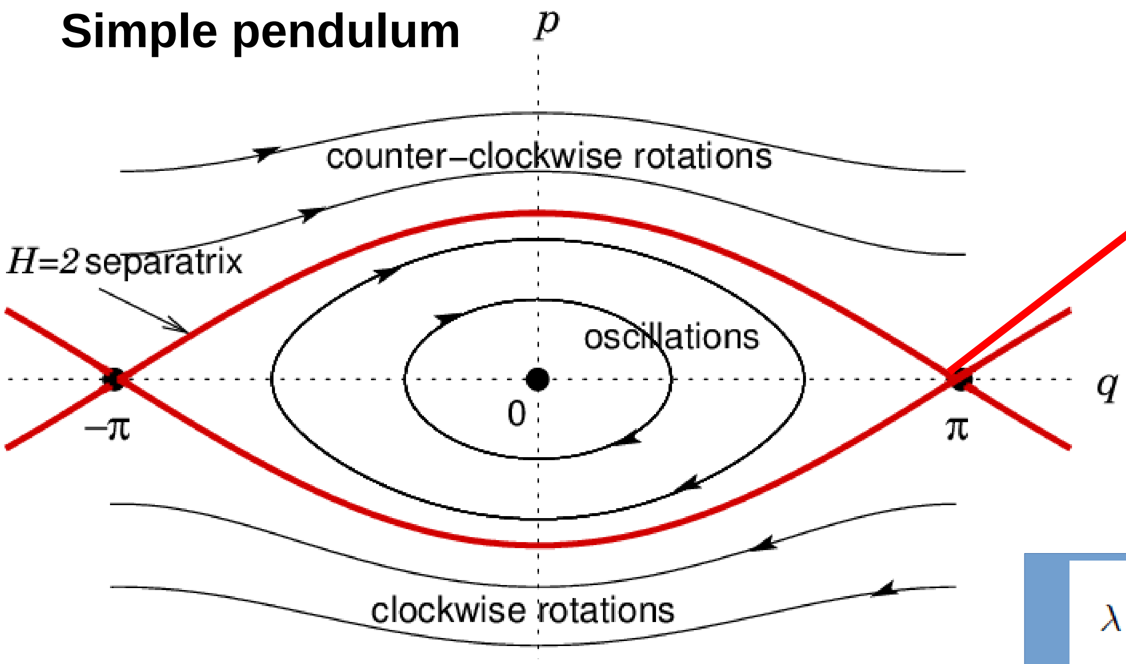


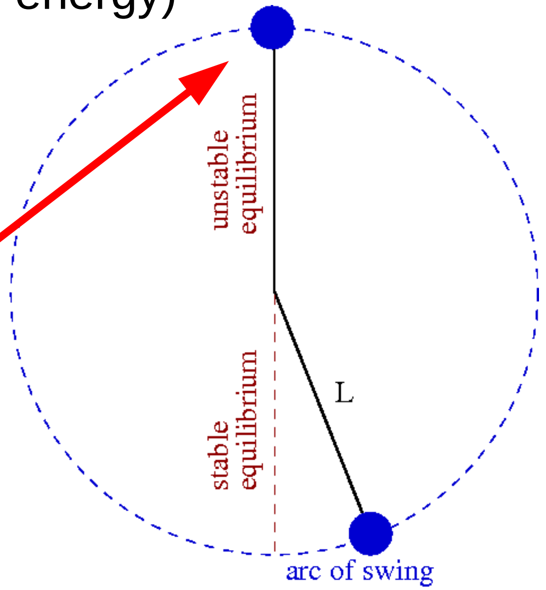


1st derivative of the density of states

Important fact: positive Lyapunov exponents also in regular systems, associated to hyperbolic unstable fixed points (critical ESQPT energy)

Simple pendulum





$$H(\theta, p_\theta) = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta)$$

$$A = D_x\mathcal{F}(x) \doteq D_x\mathcal{F}(x_0)$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \|e^{At}\|$$

$$\lambda = \lambda_{\max}(A) = \max_i \mathrm{Re}\,(\lambda_i)$$

$$\lambda_\uparrow = \omega = \sqrt{g/l}$$

The separatrix, is a measure zero set in the whole phase space, but...

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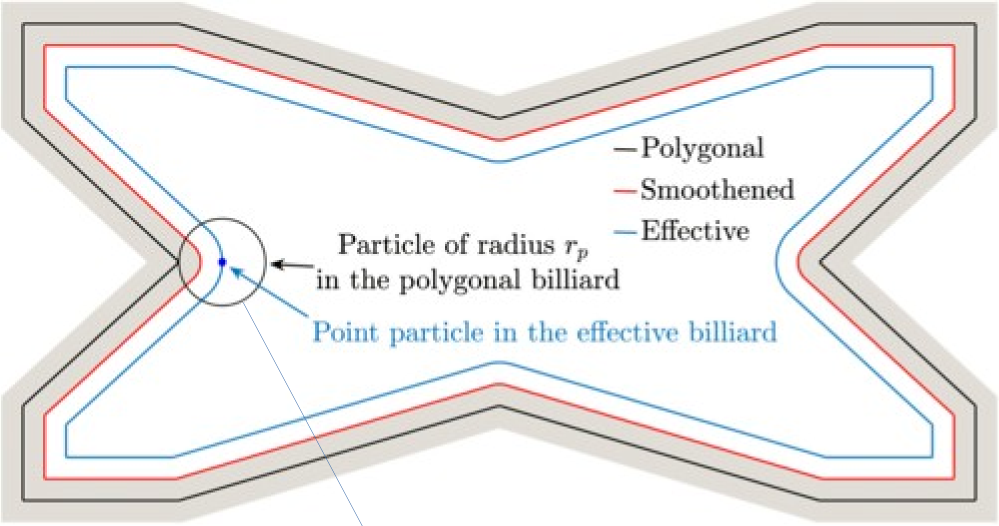
Early-Time Exponential Instabilities in Nonchaotic Quantum Systems

Efim B. Rozenbaum, Leonid A. Bunimovich, and Victor Galitski
Phys. Rev. Lett. **125**, 014101 – Published 1 July 2020

“Certain non-convex polygonal billiards, whose classical Lyapunov exponents are always zero, demonstrate a Lyapunov-like exponential growth of OTOC at early times with an h dependent Lyapunov rate”



$$r_p = \sigma\sqrt{\hbar_{\mathrm{eff}}/2}$$

An extended disk of finite radius (wave function) moves in an effective billiard with positive Lyapunov exponent.

Some results for the Lipkin-Meshkov-Glick model and Dicke model

PHYSICAL REVIEW E
covering statistical, nonlinear, biological, and soft matter physics

Rapid Communication

Positive quantum Lyapunov exponents in experimental systems with a regular classical limit

Saúl Pilatowsky-Cameo, Jorge Chávez-Carlos, Miguel A. Bastarrachea-Magnani, Pavel Stránský, Sergio Lerma-Hernández, Lea F. Santos, and Jorge G. Hirsch

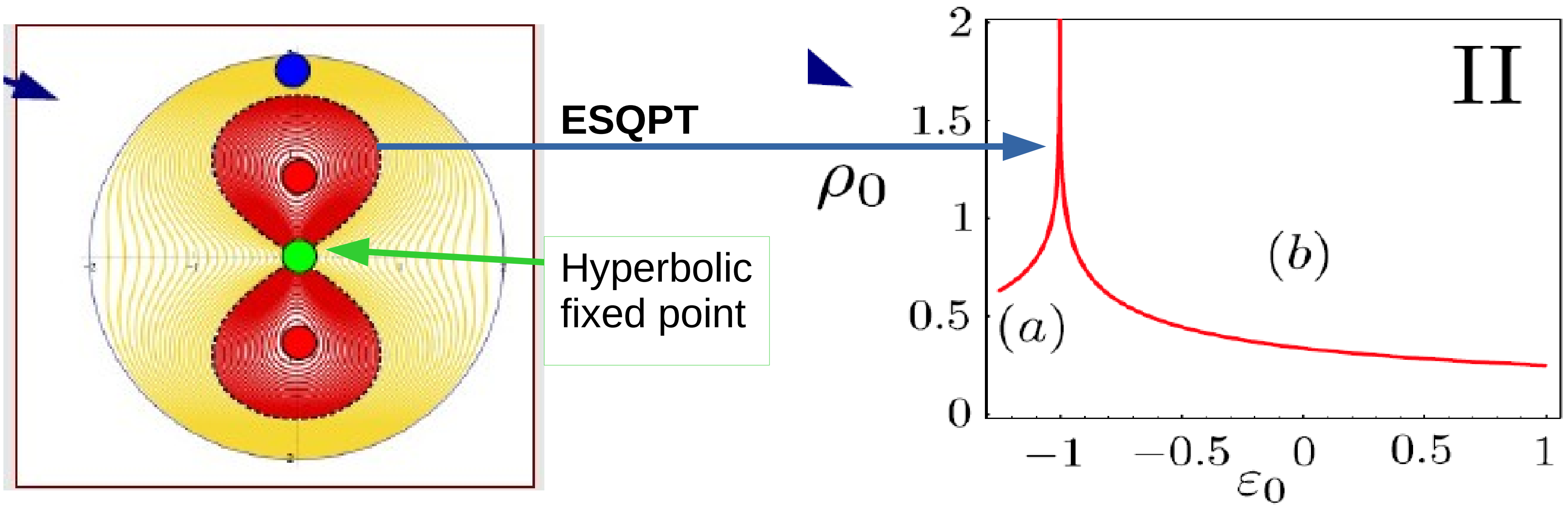
Phys. Rev. E **101**, 010202(R) – Published 22 January 2020

Simpler one degree-of-freedom model: **Lipkin-Meshkov-Glick model**

$$\hat{H}_{\mathrm{LMG}} = \Omega \hat{J}_z + \frac{2\xi}{N}\hat{J}_x^2$$

Classical Phase space

Density of states



$$H_{\mathrm{LMG}}(Q,P) = \frac{\Omega}{2}(Q^2 + P^2) - \Omega + \xi\left(Q^2 - \frac{Q^2P^2}{4} - \frac{Q^4}{4}\right)$$

Calculating classical and quantum Lyapunov exponent at ESQPT energy

FOTOC

$$\sigma_Q^2(t) + \sigma_P^2(t) \sim e^{2\Lambda t}$$

Variance of quantum operators Q and P

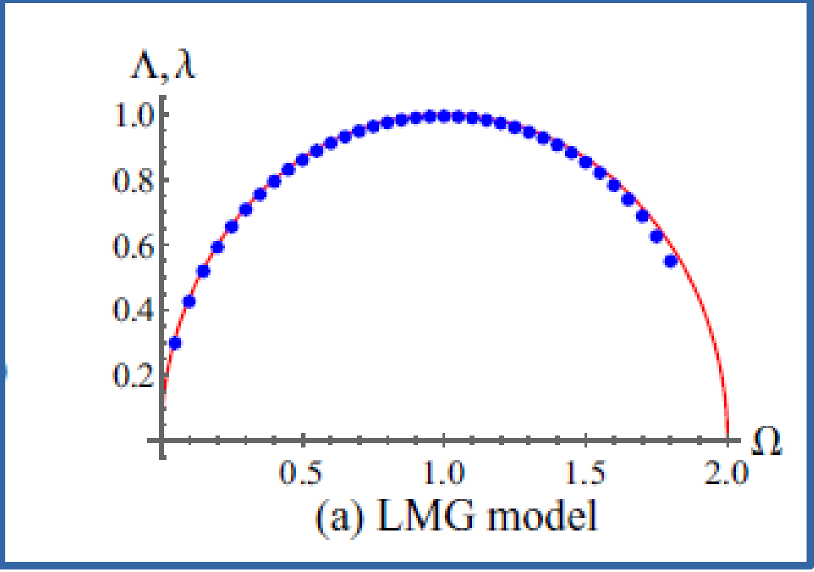
$$|\Psi_0\rangle = (1+|z|^2)^{-j} e^{z\hat{J}_+}|j,-j\rangle$$

initial coherent states centered at the hyperbolic unstable fixed point (z=0)

$$z = (Q - iP)/\sqrt{4 - (Q^2 + P^2)}$$

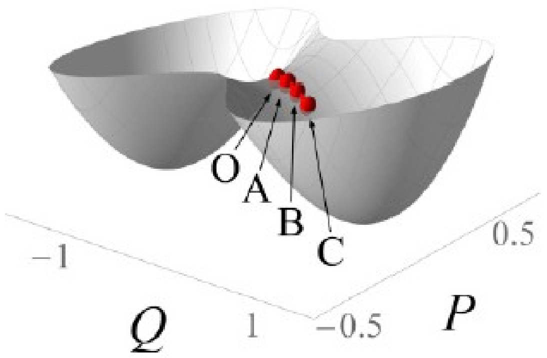
Classically...

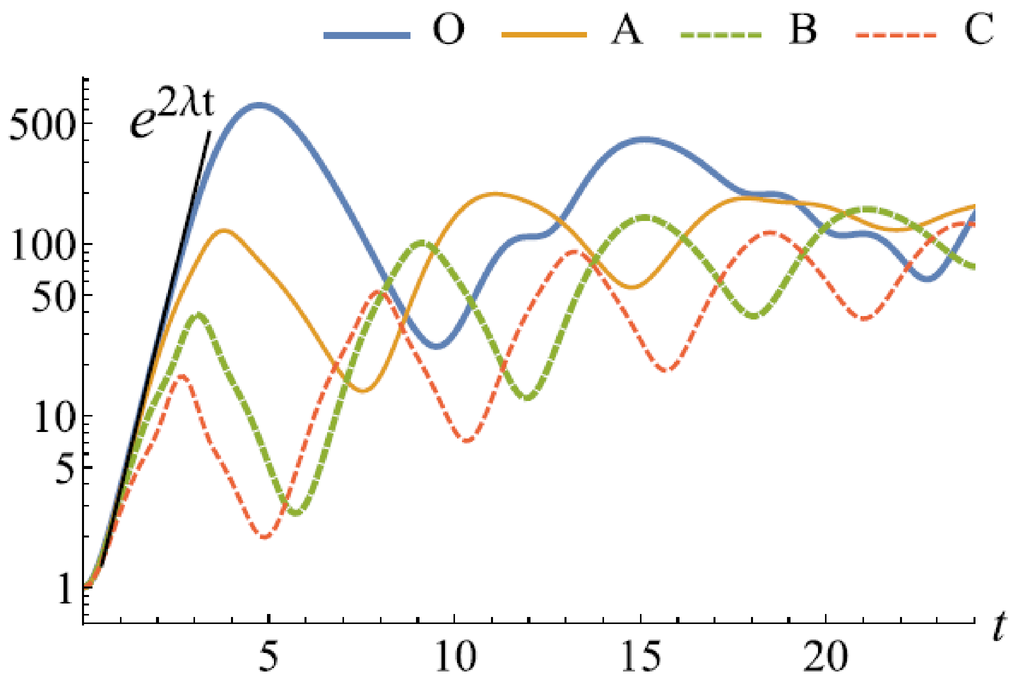
$$\lambda = \begin{cases} 0 & \text{if } \Omega \geq -2\xi \\ \sqrt{-(\Omega^2 + 2\Omega\xi)} & \text{if } \Omega < -2\xi \end{cases}$$



Perfect agreement

If we move the initial coherent state slightly away from the unstable fixed point

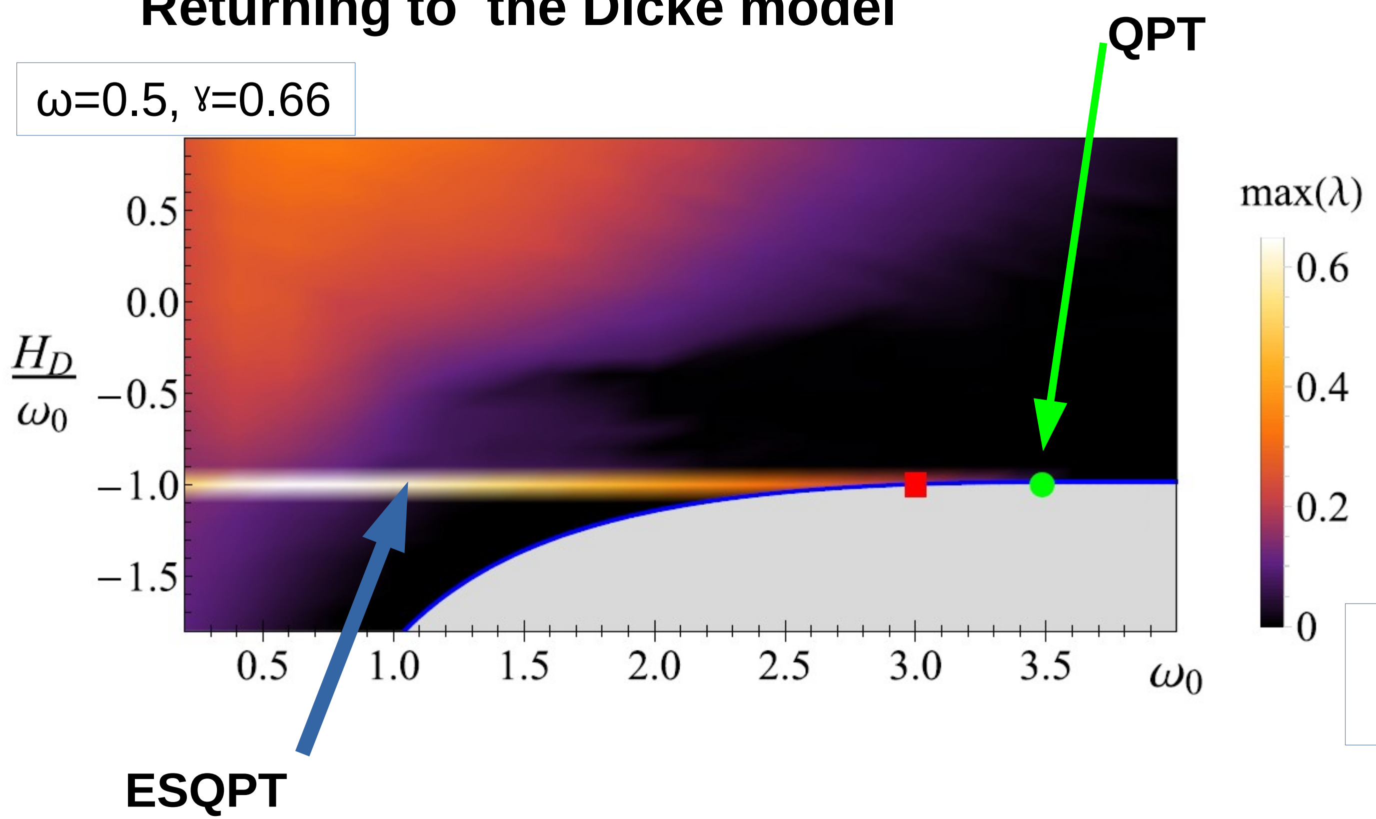




The initial exponential growth given by the classical Lyapunov exponent persists, though for a shorter time interval as we move the coherent state away from the unstable point.

Returning to the Dicke model

ω=0.5, ɣ=0.66



Unstable hyperbolic fixed point:

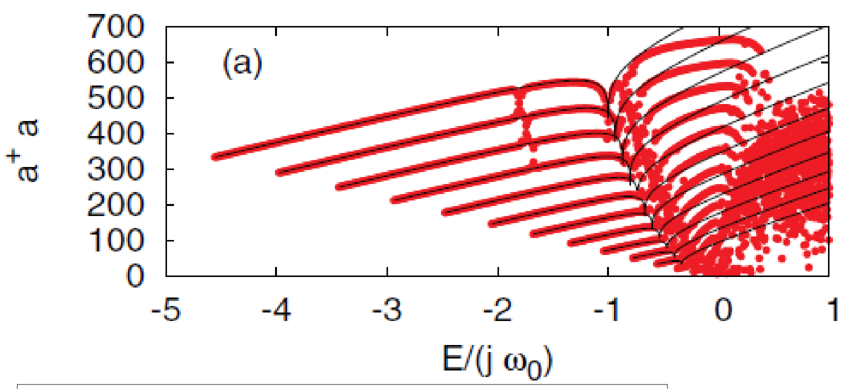
-ESQPT

-Positive classical λ, analytically evaluable

$$\lambda = \frac{1}{\sqrt{2}}\sqrt{-(\omega^2+\omega_0^2)+\sqrt{(\omega^2-\omega_0^2)^2+16\gamma^2\omega\omega_0}}.$$

For **2<ω_o<3.5** the ESQPT is surrounded by regular dynamics (λ=0).
(Fast-slow dynamical separability of atomic and bosonic degrees of freedom)

For $\omega_o \gg \omega$ the Dicke Hamiltonian is adiabatically separable

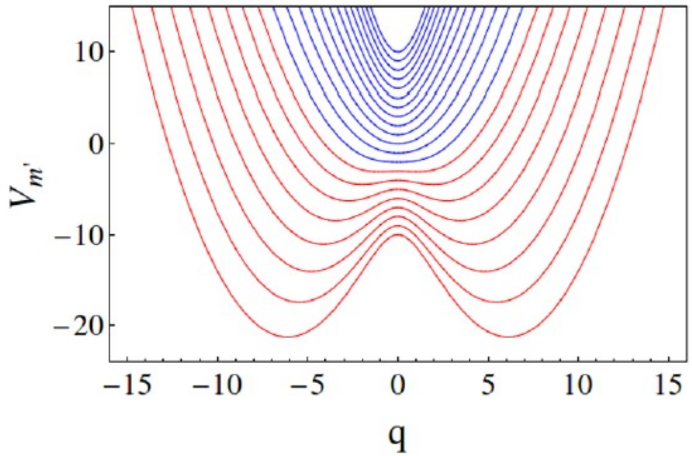


Sequence of ESQPTs

$$H(p,q) = \frac{\omega}{2}\left(p^2+q^2\right) + \omega_0\sqrt{1+f^2\frac{\omega}{\omega_0}\frac{q^2}{j}}\,\hat{J}_{z'}$$

$$\omega_0\hat{J}_z + \frac{2\gamma}{\sqrt{j}}q\hat{J}_x = \sqrt{\omega_0^2+\left(\frac{2\gamma q}{\sqrt{j}}\right)^2}\,\hat{J}_{z'}$$

$$E_{m'}(p,q) = \frac{\omega}{2}\left(p^2+q^2\right) + \omega_0\sqrt{1+f^2\frac{\omega}{\omega_0}\frac{q^2}{j}}\,m'$$



Effective potentials for the slow bosonic variables

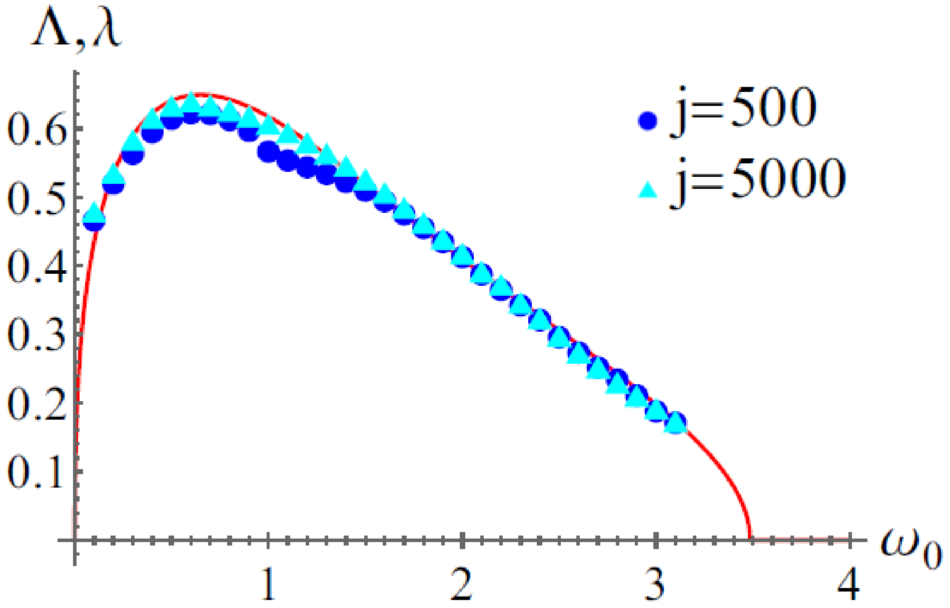
$$V_{m'}(q) = \frac{\omega}{2}q^2 + \omega_o\sqrt{1+\frac{\omega}{\omega_o}\left(\frac{f\,q}{\sqrt{j}}\right)^2}\,m'.$$

Approximated integrability of the Dicke model
A Relaño, MA Bastarrachea, S Lerma EPL 116 50005 (2016)

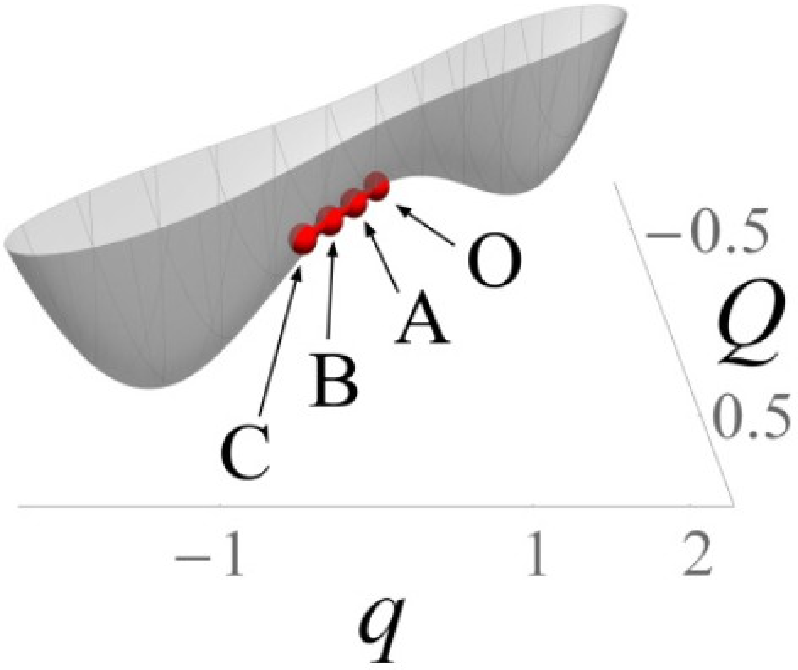
Calculating the FOTOC in the Dicke model, for initial coherent states centered in the hyperbolic fixed point (z=α=0).

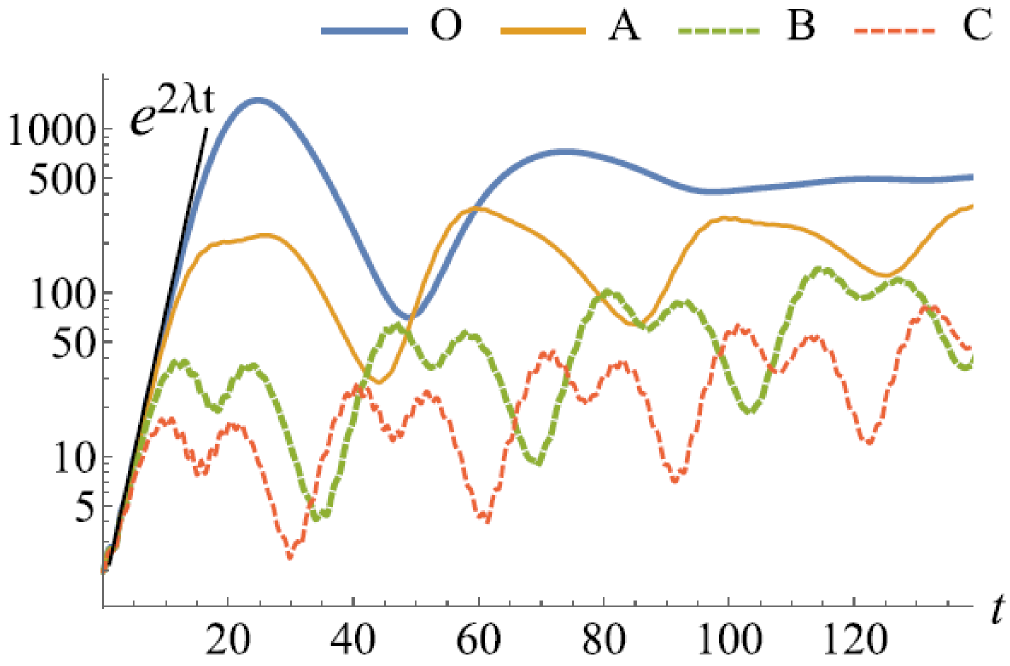
$$|\Psi_0\rangle = |z\rangle\,|\alpha\rangle$$

$$\sigma_Q^2(t)+\sigma_P^2(t)+\sigma_q^2(t)+\sigma_p^2(t) \sim e^{2\Lambda t}$$



Again, very good agreement.





Again, if we move the initial coherent state slightly away from the unstable fixed point, the exponential growth given by the Lyapunov exponent persists.

But... all the previous results are classical !!

The temporal dependence of observables can be approximated by the Truncated Wigner Approximation (TWA)

$$\langle \hat{x}_i^n(t)\rangle = \int W(\boldsymbol{x},t) x_i^n \mathrm{d}\boldsymbol{x}. \approx \int W(\boldsymbol{x})\, x_i(t)^n \,\mathrm{d}\boldsymbol{x}$$

$\boldsymbol{x}(t)$

Classical trajectory

$W(\boldsymbol{x})$ Wigner function of the initial state
(classical Liouville distribution in phase space in the TWA)

For coherent states, simple expression for the Wigner function

bosonic

$$W_{q_0,p_0}(q,p) = \frac{j}{\pi} e^{-j\Delta^2}$$

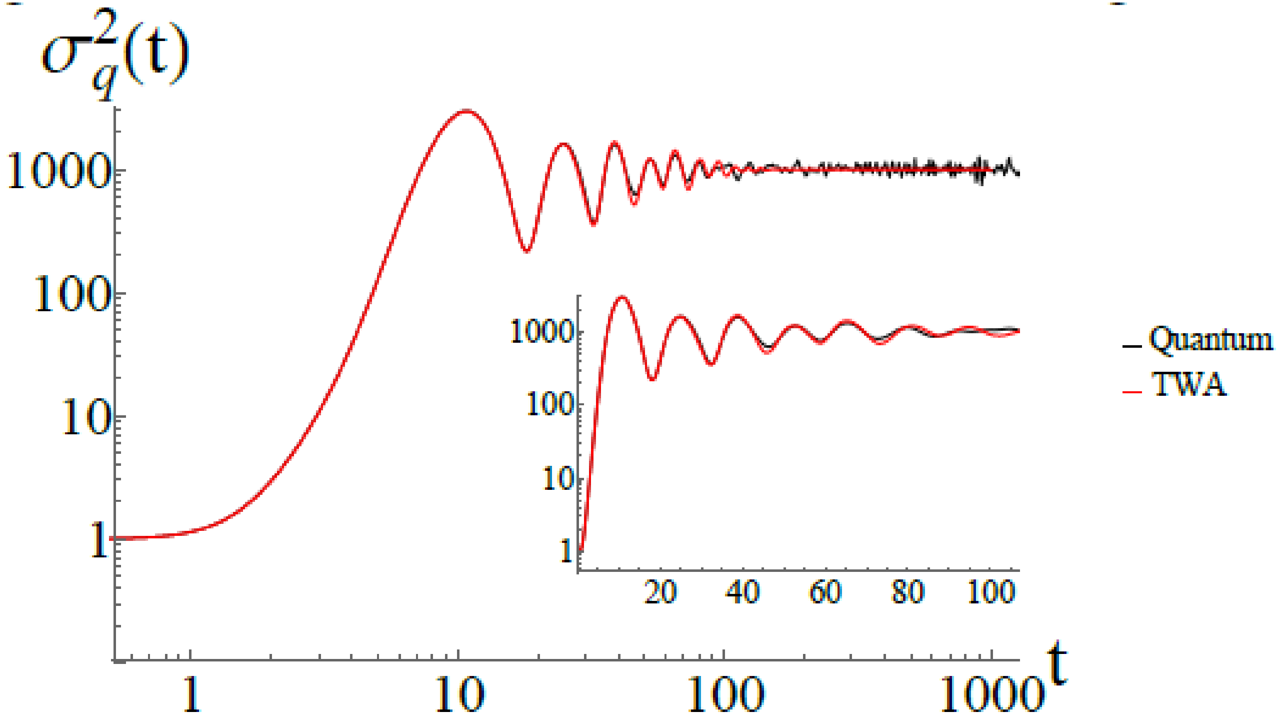
$$\Delta = \sqrt{(q-q_0)^2 + (p-p_0)^2}$$

atomic

$$W_{\theta_0,\phi_0}(\theta,\phi) \approx \frac{j}{\pi} e^{-j\Theta^2}$$

$$\cos\Theta = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)$$

Dicke model result for a coherent sate centered at the hyperbolic unstable fixed point with ESQPT energy

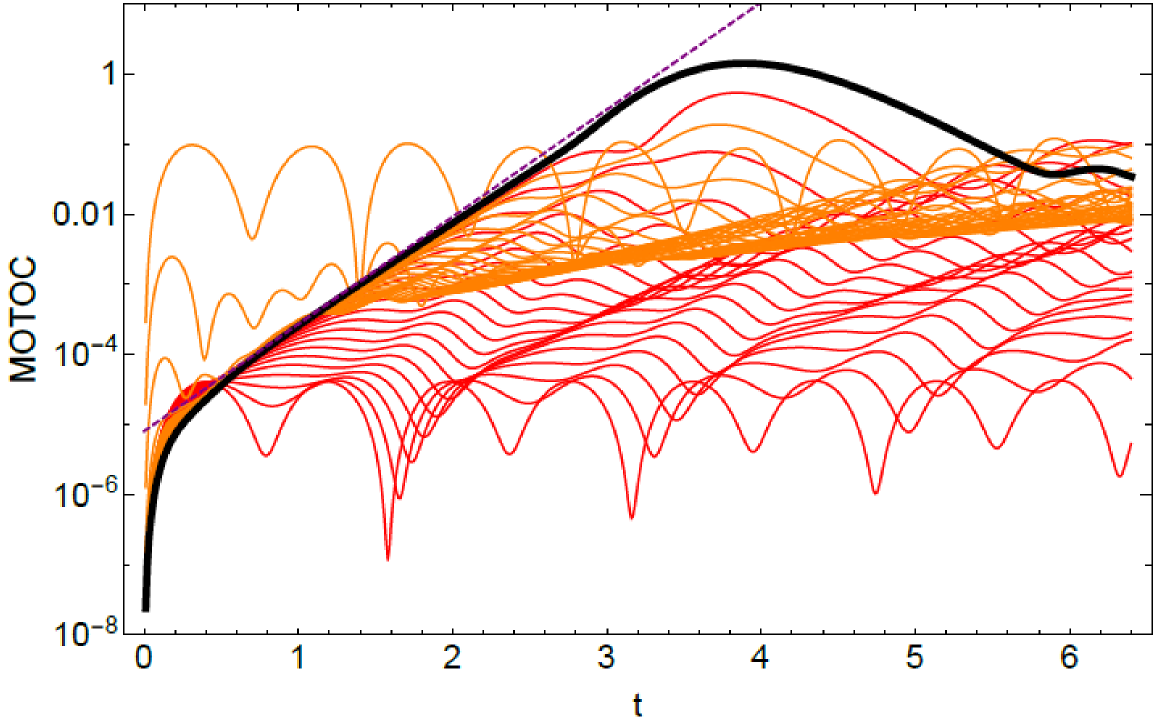


(a)

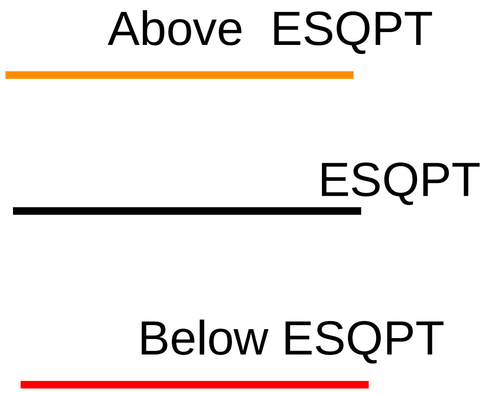
The exact quantum and classical (TWA approximated) variance of the observables coinicide at short temporal scales

MOTOC in the Lipkin-Meshkov-Glick model (unpublished)

$$C_k^{QQ}(t) = -\langle E_k|[\hat{Q}(t), \hat{Q}_o]^2|E_k\rangle$$



Initial growth of the MOTOC for Hamiltonian eigenstates of the LMG model



Is this behaviour describable by the TWA?

Analysis beyond the classical approximation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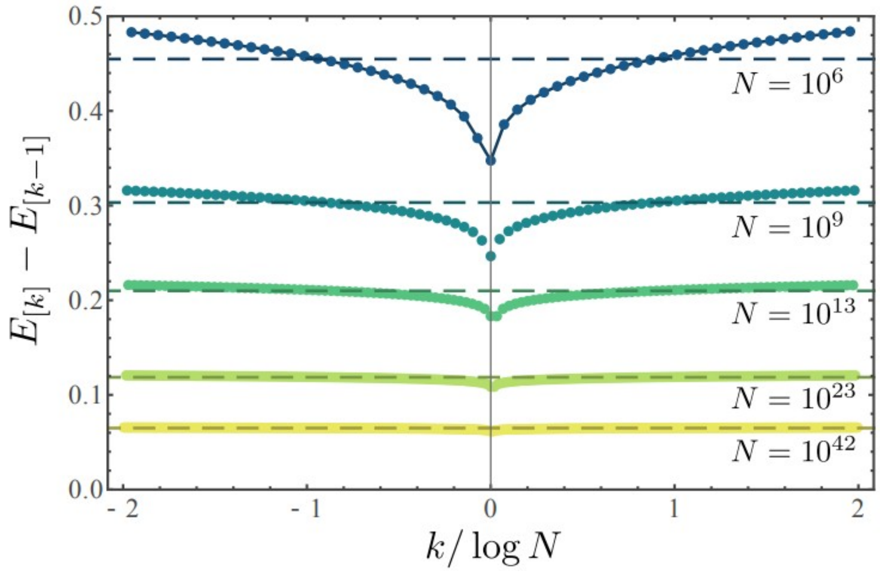
Reversible Quantum Information Spreading in Many-Body Systems near Criticality

Quirin Hummel, Benjamin Geiger, Juan Diego Urbina, and Klaus Richter

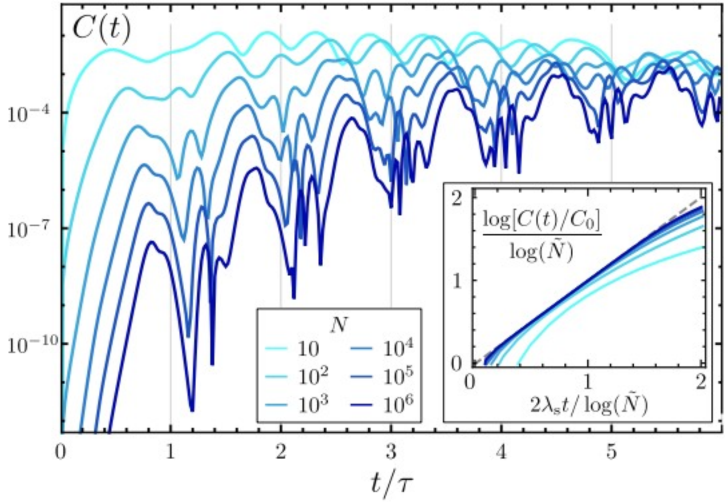
Phys. Rev. Lett. **123**, 160401 – Published 15 October 2019

$$\bar{\varrho}(E) = \frac{-1}{2\pi\lambda} \log\left(\frac{|E - E_{\rm sep}|}{\tilde{N}}\right) + \mathcal{O}(1)$$

Bunch of levels accumulating at the ESQPT energy become equally spaced as N → ∞



yielding periodic unscramblings at long temporal scales (larger than T_H)



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Editors' Suggestion

Does Scrambling Equal Chaos?

Tianrui Xu, Thomas Scaffidi, and Xiangyu Cao
Phys. Rev. Lett. **124**, 140602 – Published 7 April 2020

"... exponential growth of out-of-time order correlators (OTOCs),... can simply result from the presence of unstable fixed points in phase space, even in a classically integrable model."

Regular Article - Theoretical Physics | Open Access | Published: 13 November 2020

Exponential growth of out-of-time-order correlator without chaos: inverted harmonic oscillator

Koji Hashimoto, Kyoung-Bum Huh, Keun-Young Kim & Ryota Watanabe

Journal of High Energy Physics **2020**, Article number: 68 (2020) | Cite this article

"...exponential growth of the thermal OTOC does not necessarily mean chaos when the potential includes a local maximum."

Conclusions:

-Initial exponential growth of OTOCs associated with hyperbolic unstable fixed points and its correpondig separatrices, even for non-chaotic systems.

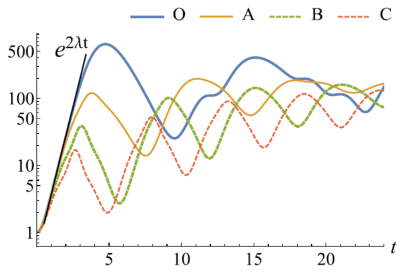
-The exponential growth of OTOCs is a reliable indicator of a kind of Excited-state-quantum phase transitions.

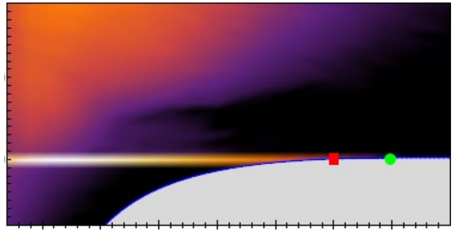
-Confirmed explicitly by our group for the LMG and Dicke models, and also for other models by other groups.

- The exponential growth, since it happens before the Ehrenfest time, is a classical result describable by the Truncated Wigner Approximation (TWA).

-As in classical mechanics, a positive Lyapunov exponent does not necessarily imply chaos.

Thank you!





Huelva, 2021