# A semiclassical approach to scrambling and revival times around criticallity

## The Huelva sessions on ESQPTs

Benjamin Geiger, Quirin Hummel, Juan-Diego Urbina, and Klaus Richter

(Regenbsurg, Liege)

April the 30th, 2021

First disclaimer:

- This talk is about semiclassics like Gutzwiller's!!

Martin Gutzwiller, "Chaos in Classical and Quantum Physics", Springer

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- semiclassical methods are asymptotic and therefore non-perturbative in $\hbar$

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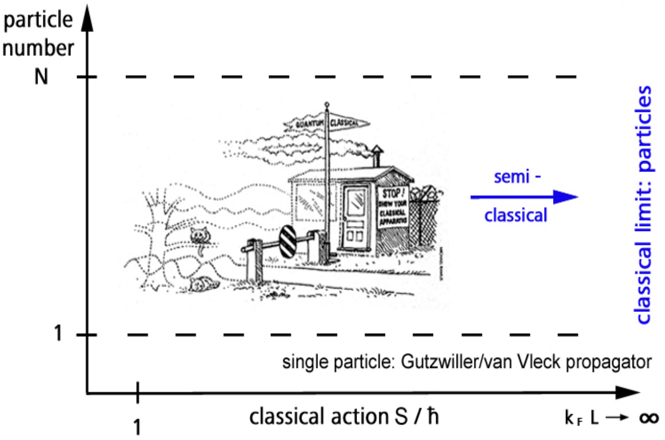
# Life at the border...



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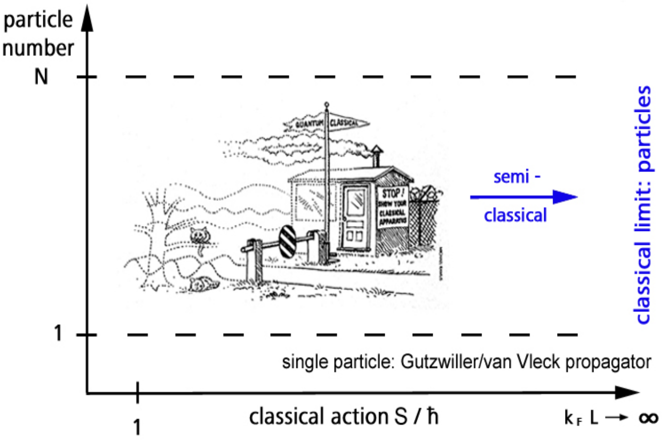


$N = 1, \hbar \to 0$ and decoherence $\to 0$: Classical Particle

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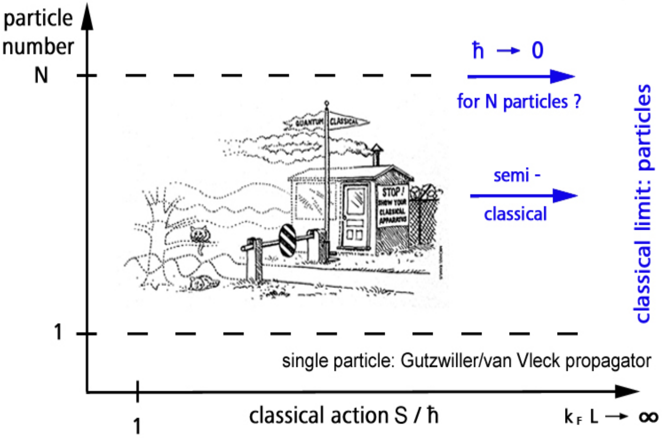

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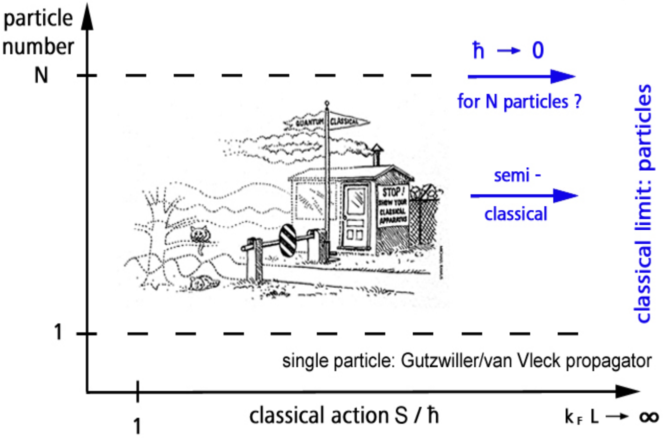


Finite $N$, $\hbar \to 0$ and decoherence $\to 0$: Classical Particles

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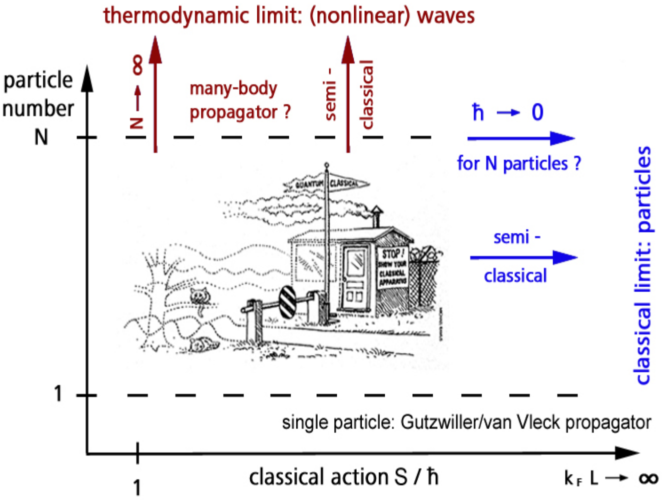

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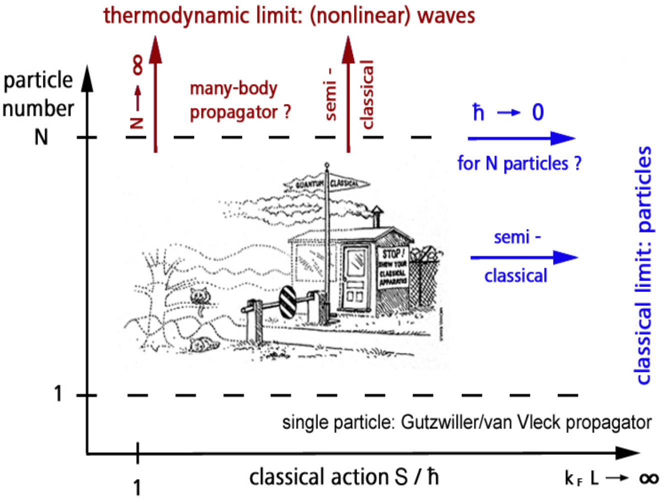


$N \to \infty$ and decoherence $\to 0$: Classical Fields

# Life at the border... can be quite singular!

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quantum($S$)

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$$\text{quantum}(S)$$
$$\neq$$

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Interference is missing

$$\mathrm{e}^{iS/\hbar}, \mathrm{e}^{iNR}$$

Non-perturbative! Example: discreteness!!!

# The transition probability

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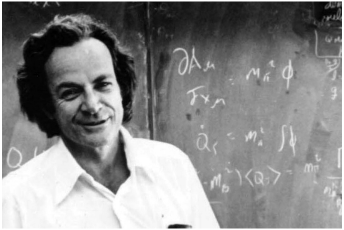

Everything starts with
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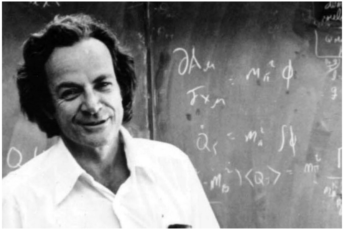

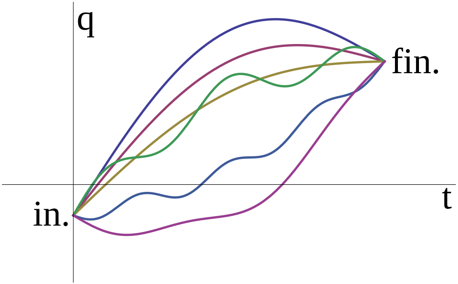


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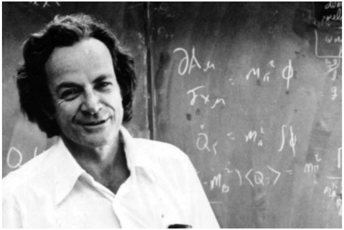
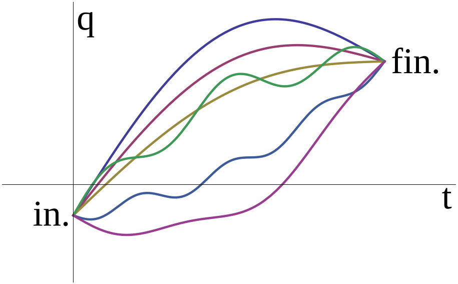
# The transition probability

Feynman path integral



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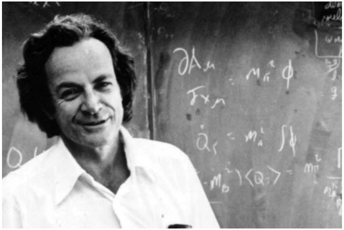

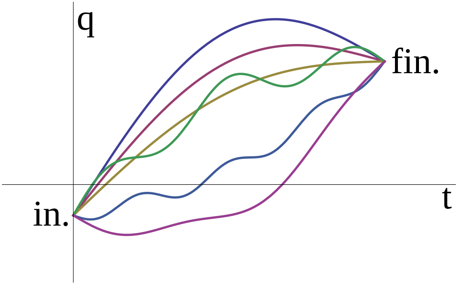


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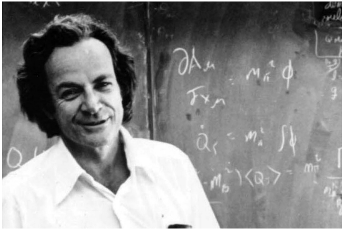

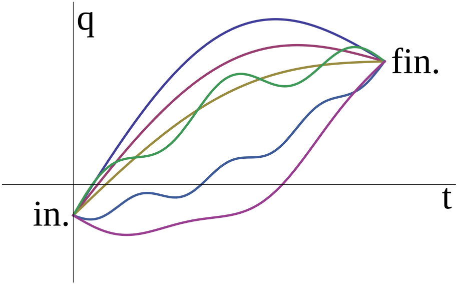


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Where are the classical paths?, can we use them?

# The semiclassical approximation ($R[q(t)] \gg \hbar$)

$$\int \mathcal{D}[q(t)]\mathrm{e}^{\frac{i}{\hbar}R[q(t)]} \simeq \sum_\gamma \sqrt{W_\gamma}\mathrm{e}^{\frac{i}{\hbar}R_\gamma + i\frac{\pi}{4}\mu_\gamma}$$

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- 1930's
- Starts from WKB
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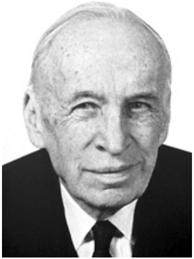

John H. van Vleck

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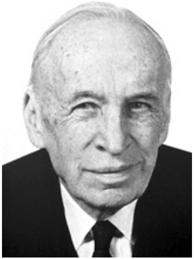

John H. van Vleck

Martin Gutzwiller

- 1970's
- Starts from Feynman
- Short and large times $\mu$

# Crash course on semiclassics (a bit technical)

Start with an action $R[q(t)]$ and the exact path integral

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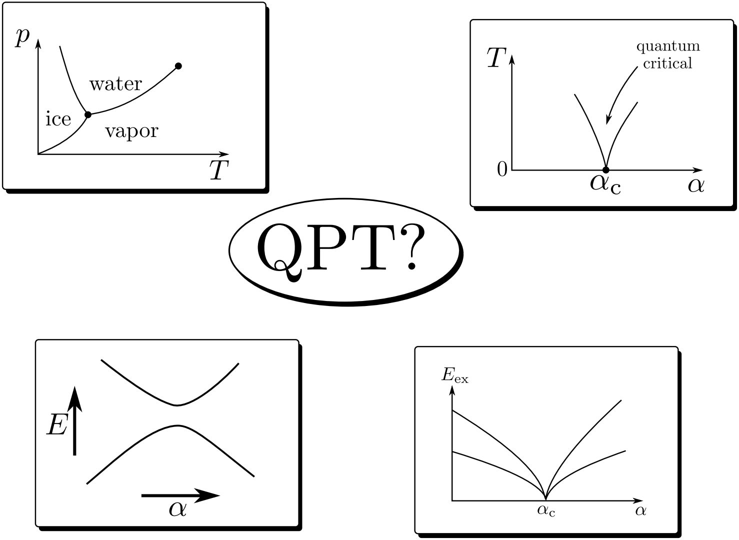

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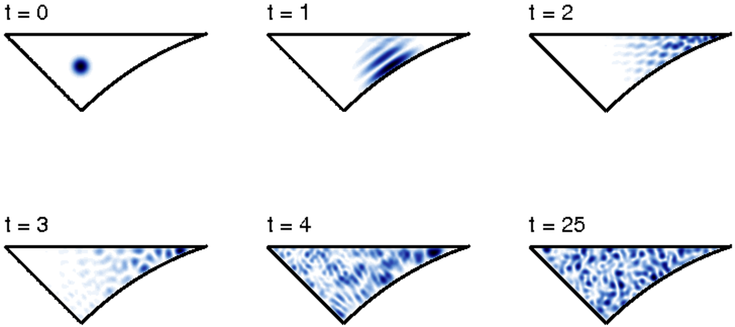

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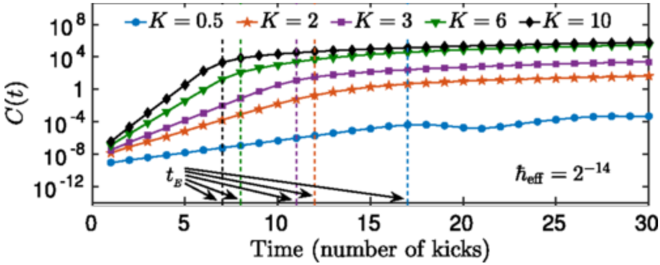


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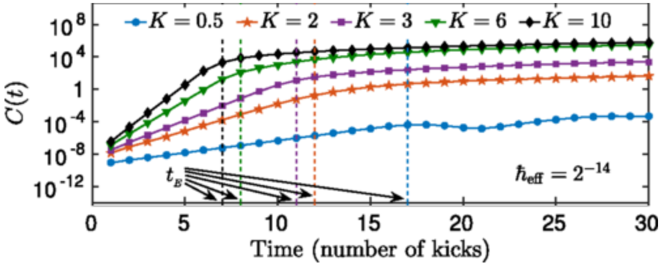


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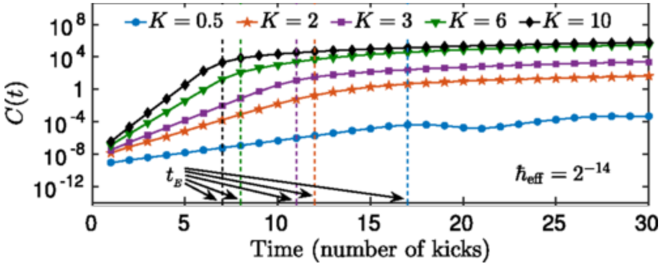


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$C(t < t_E) \sim \mathrm{e}^{2\lambda t}$: quasiclassical    Saturation: semiclassical!

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We need semiclassical methods in Many-Body Hilbert (Fock) space!!

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.... but let us take it easy....

# The Bose-Hubbard model

## States

$|\mathbf{n}\rangle = |2, 4, 3, 2, 3\rangle$

## Dynamics

$$\hat{H} = \sum_j \left[ E_j \hat{a}_j^\dagger \hat{a}_j - J \left( \hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j \right) + U \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j \right]$$

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If we know that at $t_i$ the system has occupations $n^{(i)}$,
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- Quantum states evolve as $|\phi(t_f)\rangle = \hat{U}(t_f, t_i)|\phi(t_i)\rangle$
- Transition amplitude $K(\text{fin. ; in.}) = \langle n^{(f)}|\hat{U}(t_f, t_i)|n^{(i)}\rangle$
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$K(\text{fin.}\ ;\ \text{in.})$ for Fock states?, van Vleck-Gutzwiller for fields?

# van Vleck-Gutzwiller propagator for discrete quantum fields

Engl et al PRL (2014), Phil. Trans. Roy. Soc. (2016), PRE (2015), PRA (2018) (Fermions!!)

# van Vleck-Gutzwiller propagator for discrete quantum fields

Wave equation for particles and $\hbar \to 0$
use classical trajectories

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# van Vleck-Gutzwiller propagator for discrete quantum fields

Tom

Wave equation for particles and $\hbar \to 0$
use classical trajectories

Quantum dynamics of fields and $N \to \infty$
use solutions of classical field equation

Start with a path integral and...
Do as Gutzwiller!
(easier to say than to do)

Engl et al PRL (2014), Phil. Trans. Roy. Soc. (2016), PRE (2015), PRA (2018) (Fermions!!)

# Semiclassical propagator for (Bose-) Hubbard models

$$K\left(\mathbf{n}^{(f)}, \mathbf{n}^{(i)}, t\right) = \left\langle \mathbf{n}^{(f)} \middle| \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} \middle| \mathbf{n}^{(i)} \right\rangle \approx \sum_{\gamma:\mathbf{n}^{(i)}\to\mathbf{n}^{(f)}} \mathcal{A}_\gamma \mathrm{e}^{\frac{\mathrm{i}}{\hbar} R_\gamma\left(\mathbf{n}^{(f)},\mathbf{n}^{(i)},t\right)}$$

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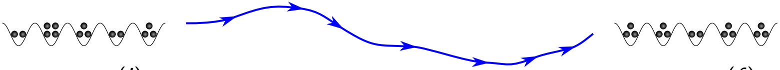
Classical trajectory $\gamma : \phi_j(s)=\sqrt{n_j(s)}\mathrm{e}^{i\theta_j(s)}$



$|\phi_j(0)|^2=n_j^{(i)}+\frac{1}{2}$ $\qquad$ $|\phi_j(t)|^2=n_j^{(f)}+\frac{1}{2}$

$$\mathrm{i}\hbar\frac{d\phi}{ds}=\frac{\partial H_{\mathrm{cl}}}{\partial\phi^*}$$

nonlinear mean-field equation (i.e GP)

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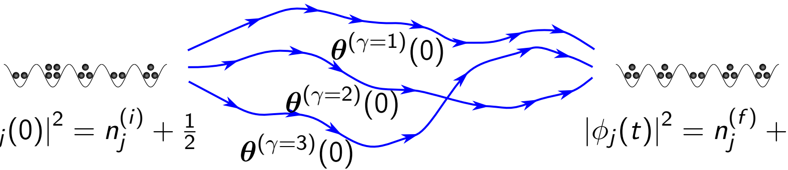


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nonlinear mean-field equation (i.e GP)

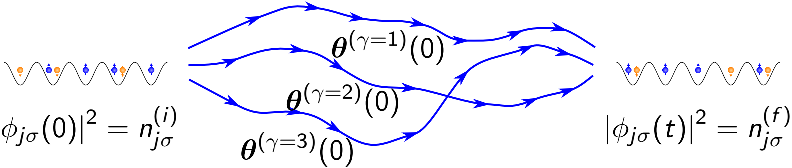
$$R_\gamma\left(\mathbf{n}^{(f)},\mathbf{n}^{(i)},t\right)=\int_0^t \mathrm{d}s\left[\hbar\boldsymbol{\theta}_\gamma(s)\cdot\dot{\mathbf{n}}_\gamma(s)-H_{\mathrm{cl}}\left(\boldsymbol{\phi}_\gamma^*(s),\boldsymbol{\phi}_\gamma(s)\right)\right]$$

# Semiclassical propagator for (Bose-) Hubbard models

path integral, $N \to \infty$, stationary phase approximation

$$K\left(\mathbf{n}^{(f)}, \mathbf{n}^{(i)}, t\right) = \left\langle \mathbf{n}^{(f)} \left| \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} \right| \mathbf{n}^{(i)} \right\rangle \overset{\downarrow}{\approx} \sum_{\gamma:\mathbf{n}^{(i)} \to \mathbf{n}^{(f)}} \mathcal{A}_\gamma \mathrm{e}^{\frac{\mathrm{i}}{\hbar} R_\gamma\left(\mathbf{n}^{(f)}, \mathbf{n}^{(i)}, t\right)}$$

Classical trajectory $\gamma : \phi_j(s) = \sqrt{n_j(s)}\mathrm{e}^{i\theta_j(s)}$

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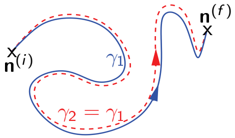


$$P(\text{fin.} \neq \text{in.}) \;=\; P_C(\text{fin.}\;;\;\text{in.})$$

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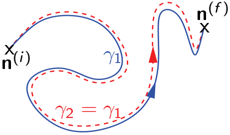


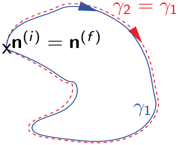


$$\begin{aligned} P(\text{fin.} \neq \text{in.}) &= P_C(\text{fin.} ; \text{in.}) \\ P(\text{fin.} = \text{in.}) &= P_C(\text{fin.} = \text{in.}) \end{aligned}$$

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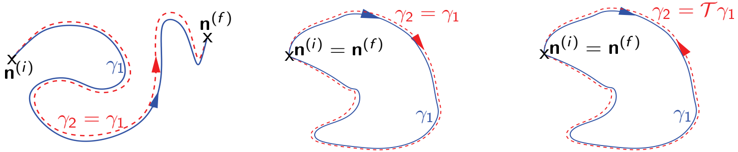


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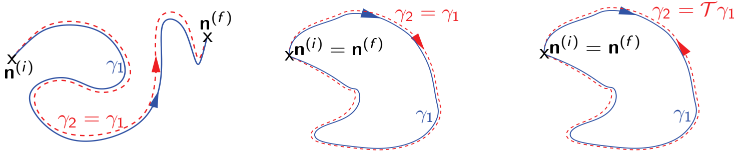
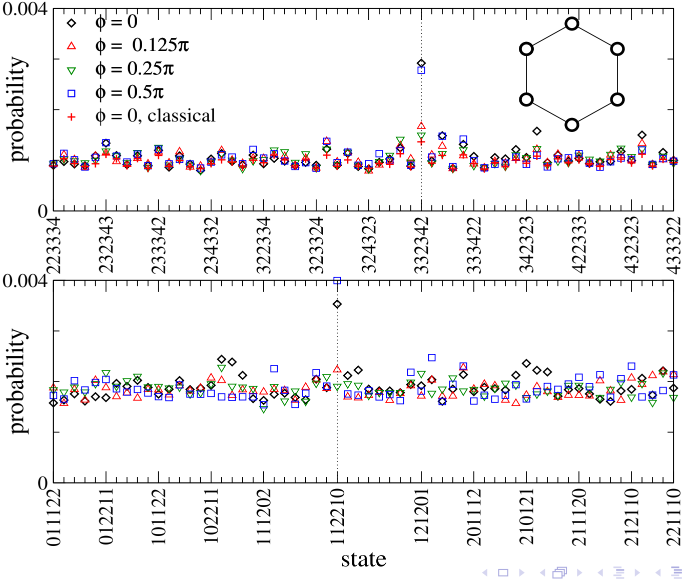



$$P(\text{fin.}\neq\text{in.})=P_C(\text{fin.};\text{in.})$$
$$P(\text{fin.}=\text{in.})=2P_C(\text{fin.}=\text{in.})$$

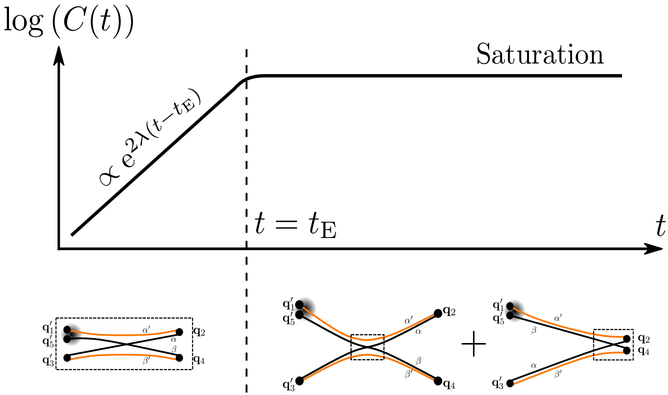
For classical (GP) invariant under $\phi(s)\to\phi^*(t-s)$ and chaotic
we predict a
coherent enhancement of the quantum probability of return!

# Checking against numerics




Engl et al PRL (2015)

# And the scrambling....?




Rammensee, JDU, Richter PRL (2018)

# Intermezzo

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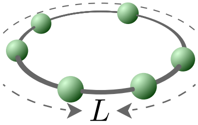
so...do we have scrambling around ESQPTs????

# The model: atractive Lieb-Liniger

Hamiltonian in second quantization:

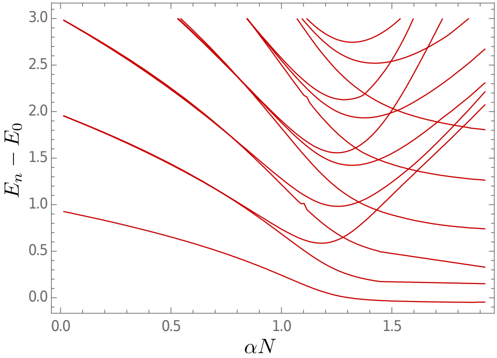
$$\hat{H} = \int_0^{2\pi} \mathrm{d}\theta \left[ \hat{\psi}^\dagger(\theta)\partial^2\hat{\psi}(\theta) - \frac{\pi\alpha}{2}\hat{\psi}^\dagger(\theta)\hat{\psi}^\dagger(\theta)\hat{\psi}(\theta)\hat{\psi}(\theta) \right]$$
$$= \sum_k k^2 \hat{a}_k^\dagger \hat{a}_k - \frac{\alpha}{4}\sum_{klmn} \delta_{k+l,m+n}\hat{a}_k^\dagger \hat{a}_l^\dagger \hat{a}_m \hat{a}_n$$

- describes one-dimensional bosonic gas with $\delta$-like short-range interactions (only $s$-wave scattering)




- model is integrable for periodic boundary conditions
  - $\rightarrow$ infinite number of conservation laws (including number + momentum conservation)
  - $\rightarrow$ look for reduced system by truncating $k$-summation

# Effect of truncation

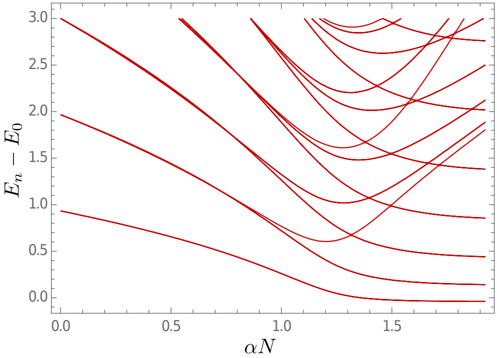



$N = 20$
$k_{\max} = \infty$

Sykes et. al.,
Phys. Rev. A76,063620

- Number and momentum conservation is not destroyed by truncation
- System is again integrable for $k_{\max} = 1$ (commonly used)
- Low-energy spectrum is quite similar, i.e. interesting properties are preserved

# Effect of truncation




$N = 20$
$k_{\max} = 2$

Sykes et. al.,
Phys. Rev. A76,063620

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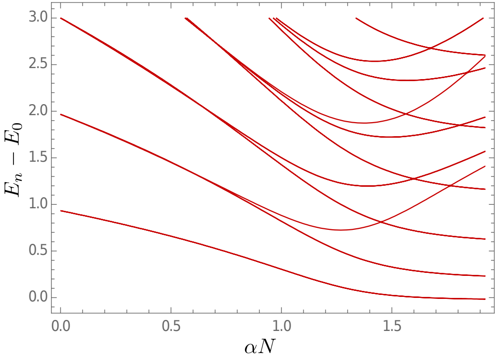


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- System is again integrable for $k_{\max} = 1$ (commonly used)
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# Scheme of SC treatment of $\hat{\mathbf{H}}_3$

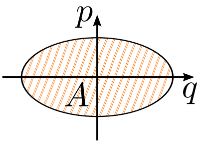
- find classical Hamiltonian by symmetrizing operators and replacing

$$\hat{a}_k \to \phi_k = \frac{1}{\sqrt{2}}(q_k + \mathrm{i}p_k) = \sqrt{n_k}\mathrm{e}^{\mathrm{i}\theta_k}$$
$$\hat{a}_k^\dagger \to \phi_k^* = \frac{1}{\sqrt{2}}(q_k - \mathrm{i}p_k) = \sqrt{n_k}\mathrm{e}^{-\mathrm{i}\theta_k}$$

- eliminate $n_{-1}$ and $n_1$ in favor of the COM

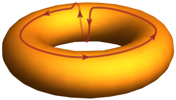
$$\tilde{N} = n_{-1} + n_0 + n_1, \qquad \tilde{L} = n_1 - n_{-1}$$

- quantize resulting Hamiltonian using torus quantization




$A = S(E, \tilde{N}, \tilde{L}) = 2\pi\hbar(n + \nu/4)$
$\nu$: Maslov index
$n = 0, 1, \ldots$

# (Re-)Quantization of $\mathbf{H_3}$



- Main difficulty: identification of the primitive orbits on the 3-Torus
- Correct quantization rules for $\tilde{N}$ and $\tilde{L}$:

$$\tilde{N} = N + \frac{3}{2}, \quad N = 0, 1, \ldots, \qquad \tilde{L} = L \in [-N, N]$$

- For the rest of the talk: $L = 0$
- Rescaled energy:

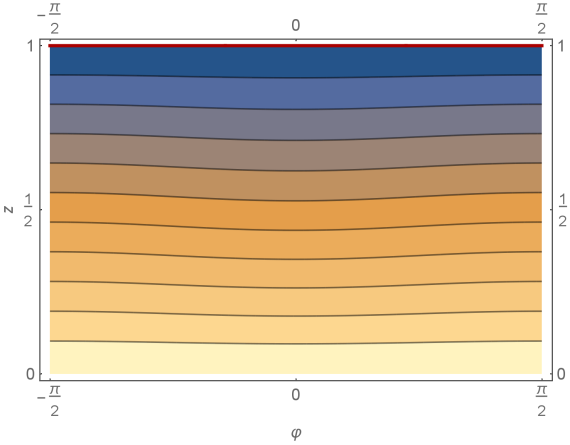
$$\omega(z, \varphi) = (1 - z) - \frac{\tilde{\alpha}}{4} \left[ \frac{(1 - z)^2}{2} + 4z(1 - z) \cos^2 \varphi \right]$$

$$z = \frac{n_0}{\tilde{N}}, \qquad \tilde{\alpha} = \tilde{N}\alpha, \qquad \omega = \frac{E}{\tilde{N}} + c(\tilde{N}, \tilde{\alpha})$$

- Rescaled Poisson bracket: $\{z, \varphi\} = \frac{1}{\tilde{N}} = \hbar_{\text{eff}}$

# Phase space structure

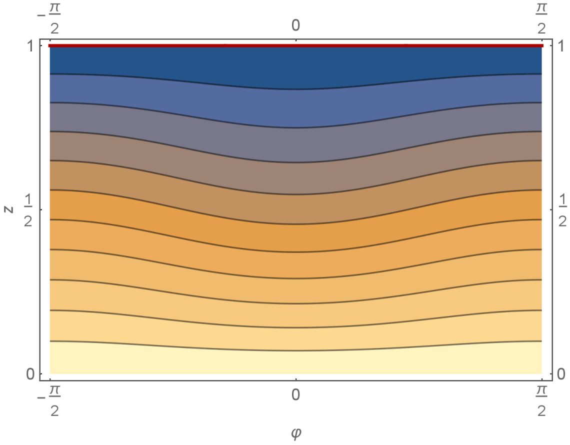
$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 0.1$

# Phase space structure

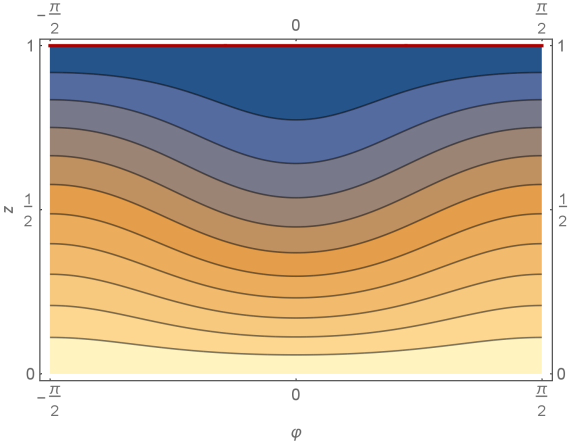
$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 0.4$

# Phase space structure

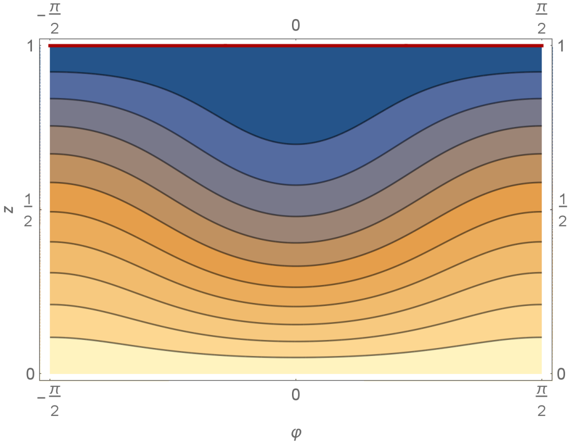
$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 0.8$

# Phase space structure

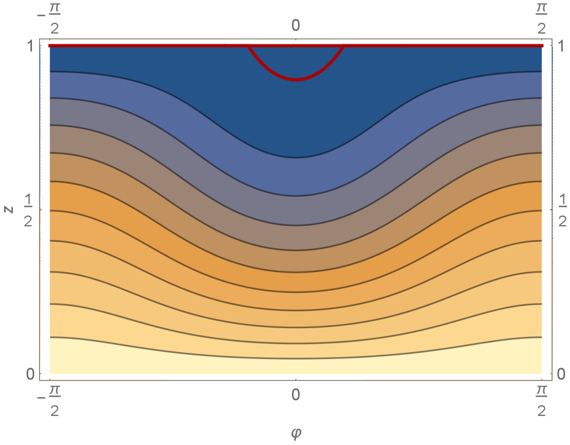
$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 1.0$

# Phase space structure

$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 1.1$

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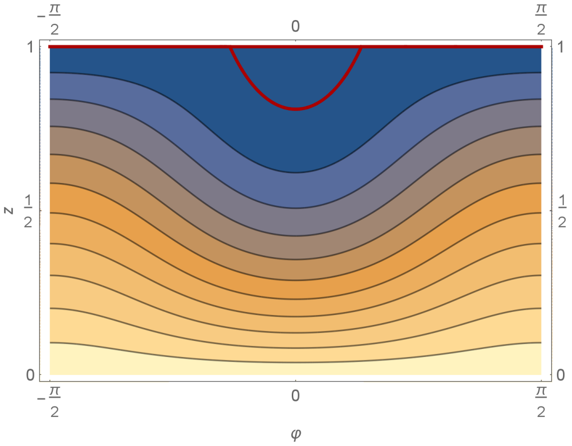
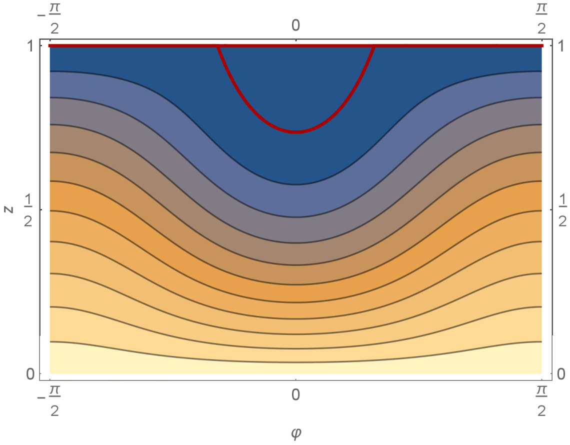



$\tilde{\alpha} = 1.2$

# Phase space structure

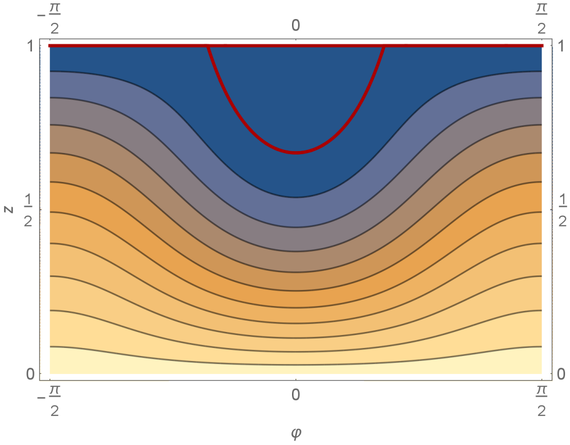
$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 1.3$

# Phase space structure

$$\omega(z,\varphi) = (1-z) - \frac{\tilde{\alpha}}{4}\left[\frac{(1-z)^2}{2} + 4z(1-z)\cos^2\varphi\right]$$




$\tilde{\alpha} = 1.4$

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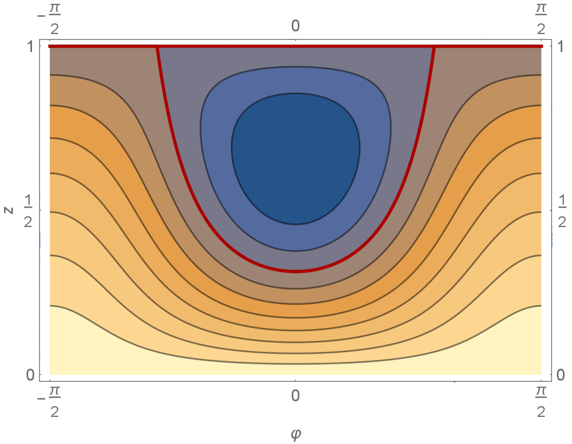
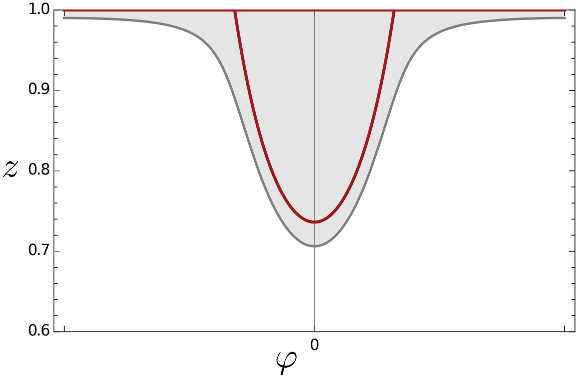



$\tilde{\alpha} = 2.0$

# Energy quantization

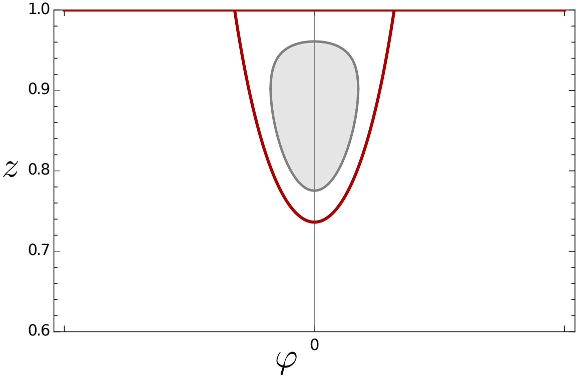
$$\frac{1}{2\pi}\oint \mathrm{d}\varphi\,(1 - z(\omega,\tilde{N})) = \frac{k+\frac{1}{2}}{\tilde{N}} = \hbar_{\mathrm{eff}}(k+\frac{1}{2})$$




$\tilde{\alpha} = 1.3$
$\omega = 0.01$

# Energy quantization

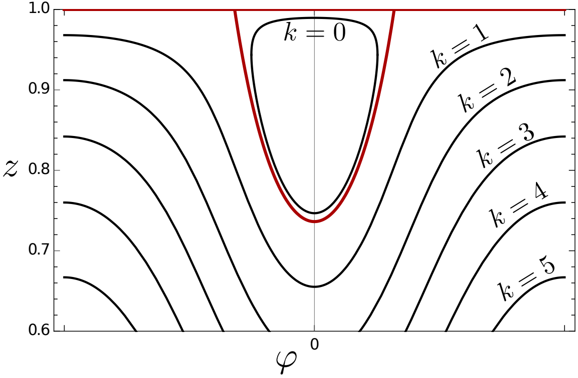
$$\frac{1}{2\pi}\oint \mathrm{d}\varphi\,(1 - z(\omega, \tilde{N})) = \frac{k + \frac{1}{2}}{\tilde{N}} = \hbar_{\text{eff}}(k + \frac{1}{2})$$




$\tilde{\alpha} = 1.3$
$\omega = -0.01$

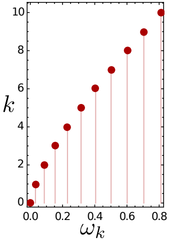
# Energy quantization

$$\frac{1}{2\pi}\oint \mathrm{d}\varphi\,(1 - z(\omega, \tilde{N})) = \frac{k+\frac{1}{2}}{\tilde{N}} = \hbar_{\rm eff}(k+\frac{1}{2})$$

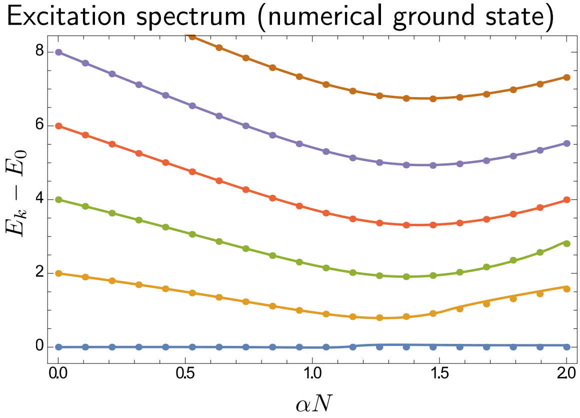



$\tilde{\alpha} = 1.3$

$N = 20$



# Comparison with exact diagonalization

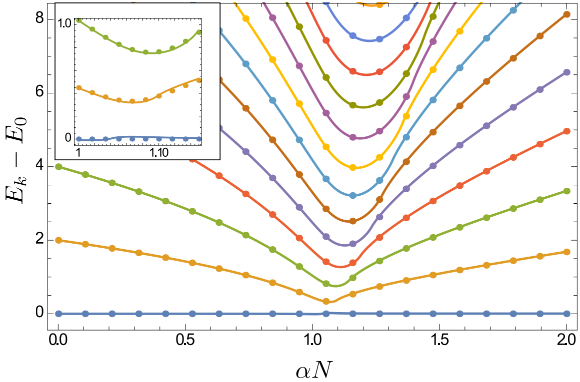



Dots: exact
Lines: SC

$N = 20$

# Comparison with exact diagonalization

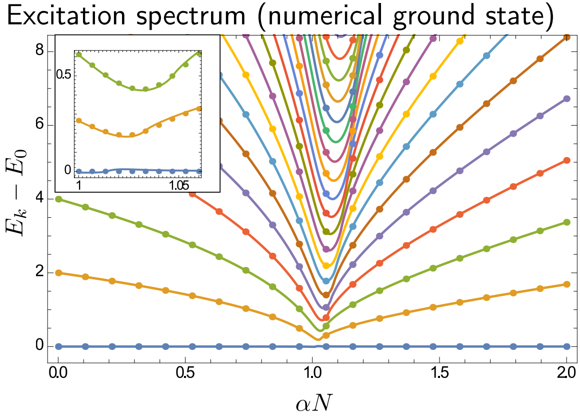
Excitation spectrum (numerical ground state)




Dots: exact
Lines: SC

$N = 200$

# Comparison with exact diagonalization

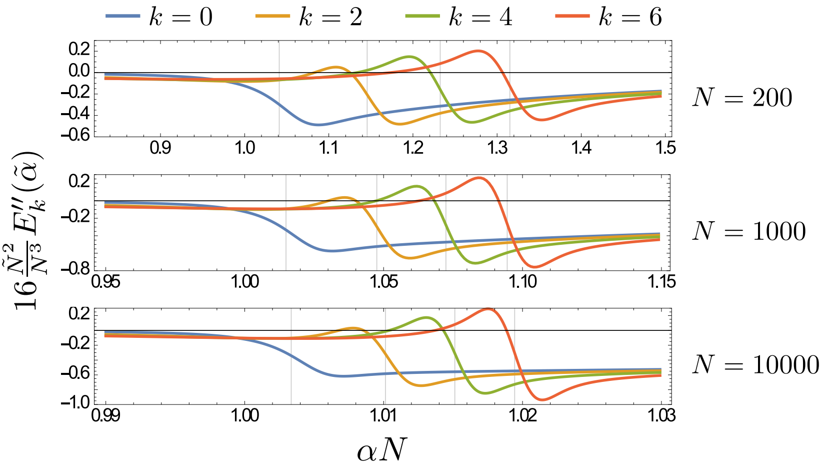



Dots: exact
Lines: SC

$N = 1000$

# Excited state quantum phase transition

A QPT of $k^{\text{th}}$ order is related to a discontinuity in the $k^{\text{th}}$ derivative of the energy $E_n$



# Excited state quantum phase transition

Interpretation of discontinuity:

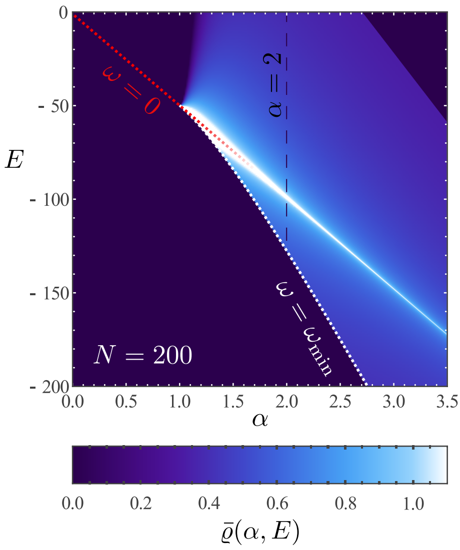
$$
\begin{aligned}
\frac{\mathrm{d}E_k}{\mathrm{d}\tilde{\alpha}} &= \langle\psi_k|\frac{\mathrm{d}\hat{H}}{\mathrm{d}\tilde{\alpha}}|\psi_k\rangle \\
&= -\frac{\pi}{2\tilde{N}}\int_0^{2\pi}\mathrm{d}\theta\ \langle\psi_k|\hat{\psi}^\dagger(\theta)\hat{\psi}^\dagger(\theta)\hat{\psi}(\theta)\hat{\psi}(\theta)|\psi_k\rangle \\
&= -\frac{\pi^2}{\tilde{N}}\langle\psi_k|\hat{\psi}^\dagger(0)\hat{\psi}^\dagger(0)\hat{\psi}(0)\hat{\psi}(0)|\psi_k\rangle \\
&= -\frac{\pi^2}{\tilde{N}}\left(\frac{N}{2\pi}\right)^2 g_2^{(k)}(\tilde{\alpha})
\end{aligned}
$$

$g_2^{(k)}$: normalized local two-point correlation of $k^{\text{th}}$ state

$\Rightarrow$ sudden increase of pair correlation at $\tilde{\alpha} = \tilde{\alpha}_{\mathrm{cr}}^{(k)} > 1$

$\Rightarrow$ bunching of particles/bound state formation

# Excited state quantum phase transition



# Mesoscopic (large-N) aspects of first excitation

- Minimum involves a vibration (ground state) and the libration closest to the separatrix
- For $N \gg 1$ this situation occurs for $\tilde{\alpha} \approx 1$, i.e. separatrix enters allowed phase space only for $\varphi \ll 1$
- Action integrals can be approximated for small angles
- Equation for the gap minimum has universal scaling:

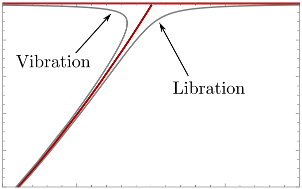
$$
\begin{aligned}
\tilde{\alpha}_{\min} =& 1 + \left( \frac{21\pi}{32 q_\infty} \right)^{\frac{2}{3}} \cdot \tilde{N}^{-\frac{2}{3}} \\
\Delta E_{\min}(\tilde{N}) =& \frac{2}{7} \left( \frac{21\pi}{32 q_\infty} \right)^{\frac{4}{3}} \Delta\mu_\infty \cdot \tilde{N}^{-\frac{1}{3}},
\end{aligned}
$$

with universal constants $q_\infty = 0.525\ldots$, $\Delta\mu_\infty = 0.953\ldots$

# Energy spacing near transitions

Fix $\tilde{\alpha} > 1$ and calculate energies near the separatrix.

- Classical orbits close to a separatrix bypass hyperbolic fixed points
- Traversal time of an orbit on the separatrix diverges logarithmically
- Largest contribution to action comes from neighborhood of the fixed points $\rightarrow$ quadratic expansion needed

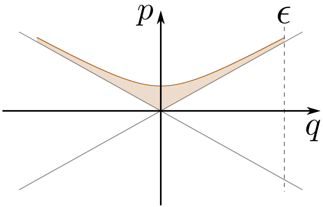


# Energy spacing near transitions

Generic hamiltonian after canonical transformation:

$$H_{\mathrm{FP}} = \frac{1}{2}((\lambda \cdot p)^2 - q^2), \qquad \lambda\text{: stability exponent}$$

$$\begin{aligned}\Delta S[E] &= |S[E] - S[0]| \\ &= \frac{1}{\lambda}\int_{-\epsilon}^{\epsilon} \mathrm{d}q \sqrt{|E| + q^2} \\ &= -\frac{1}{\lambda}|E| \log|E| + \mathcal{O}(E)\end{aligned}$$

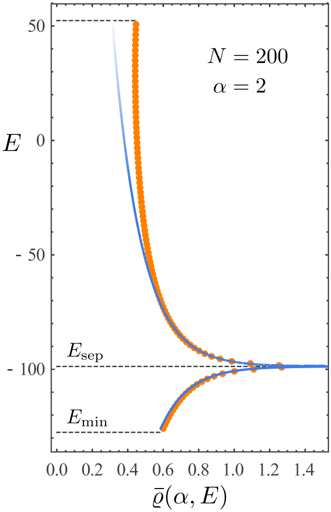



Inverting $\Delta S[E] = 2\pi\hbar(k + \mu)$ involves Lambert-W function. But for very small $E$ it yields

$$\Delta E = \frac{2\pi\hbar\lambda}{-\log(\hbar)} \overset{\text{3-site}}{=} \frac{2\pi\sqrt{\tilde{\alpha} - 1}}{\tilde{N}\log\tilde{N}}$$

# The log time scale

this we know...



# The log time scale

Large-$N$ energy scaling at the separatrix suggests time scale

$$\tau \propto \frac{2\pi\hbar_{\text{eff}}}{\Delta E} = \frac{\log(N)}{\lambda}$$

- Resembles Ehrenfest time $-\frac{1}{\lambda}\log\hbar$ in chaotic systems, where $\lambda$ is the Lyapunov exponent
- Link between chaos and instabilities in integrable systems: SCRAMBLING!!!
- Calculate scrambling time as an indicator for breakdown of classical description Geiger, JDU, Richter PRL (2021)

# Log time as scrambling time: numerical check

Scheme for numerical calculation:

- Calculate one-body density matrix

$$\rho_{ij} = \frac{1}{N} \langle \psi(t) | \hat{a}_i^\dagger \hat{a}_j | \psi(t) \rangle$$

  for the time evolved condensate

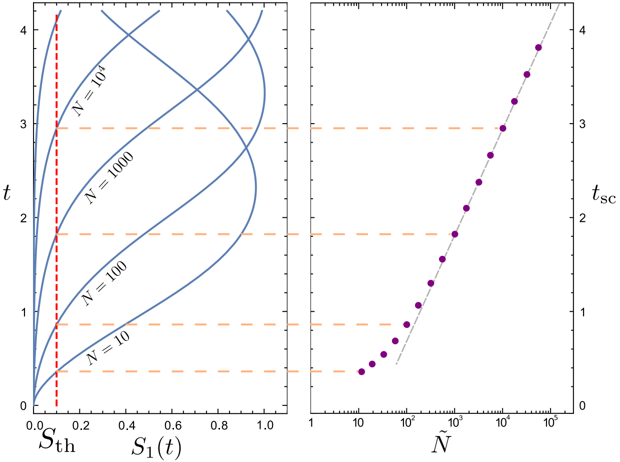
$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}t\hat{H}} \, |N\rangle$$

- Calculate von Neumann-entropy

$$S_1(t) = -\operatorname{Tr} \rho \log \rho$$

- Define scrambling time $t_\mathrm{s}$ as the time needed to reach a certain threshold value
- Do this for different particle numbers

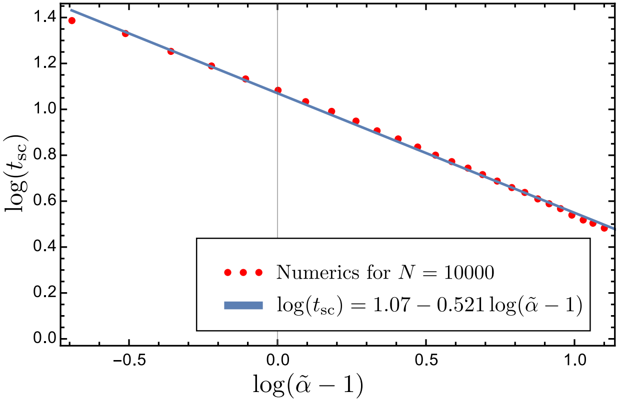
# Scrambling time (numerical results)




$t_{\mathrm{sc}} \sim \frac{\log(\tilde{N})}{\sqrt{\tilde{\alpha}-1}}$

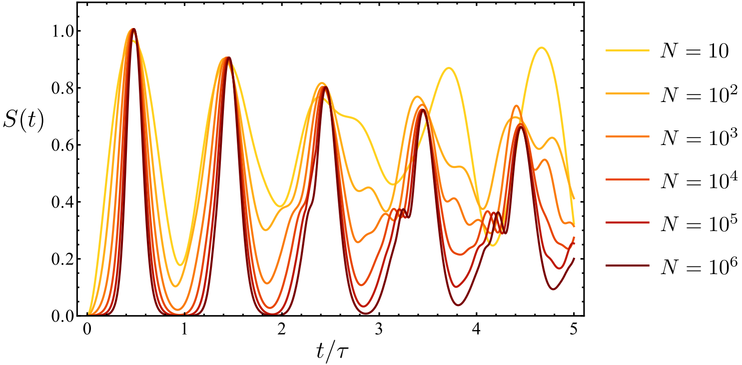
$\tilde{\alpha} = 2$
$\tilde{N}$ varying

# Scrambling time (numerical results)




$$t_{\mathrm{sc}} \sim \frac{\log(\tilde{N})}{\sqrt{\tilde{\alpha}-1}}$$

# In real time...



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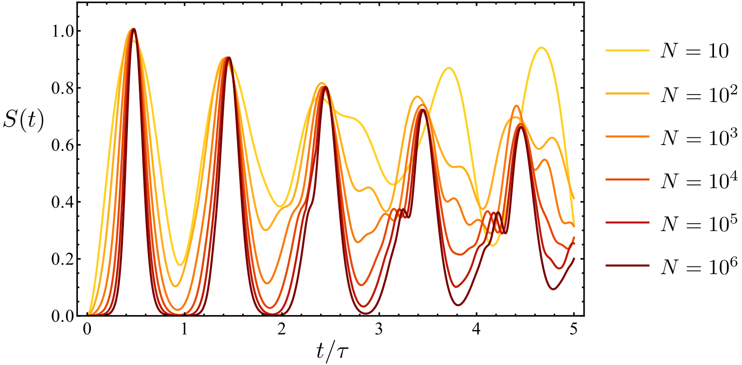


Wait a second... what are these revivals doing there?

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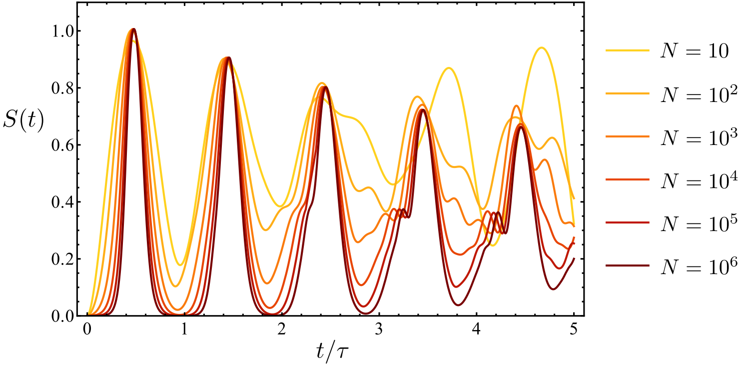
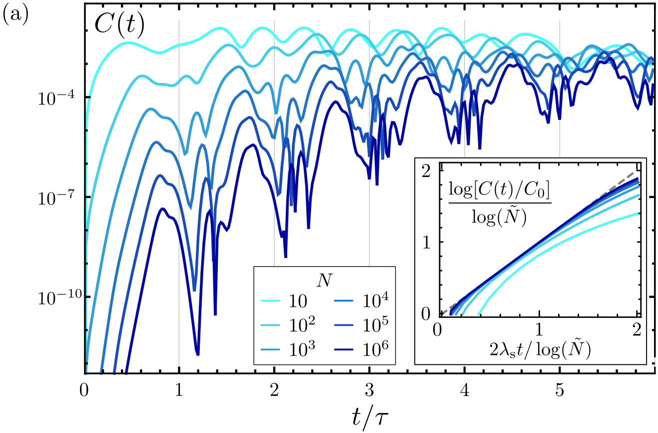



Wait a second... what are these revivals doing there?
These are NOT related with the (astronomical) recurrence times...

# What about the OTOCs???



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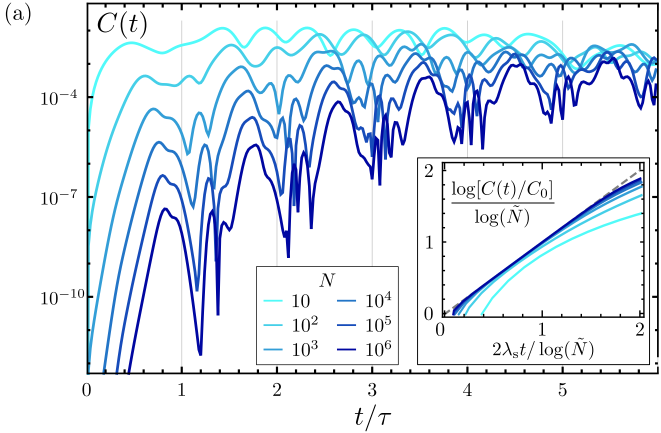


Coexistence of fast initial scrambling and long-time revivals

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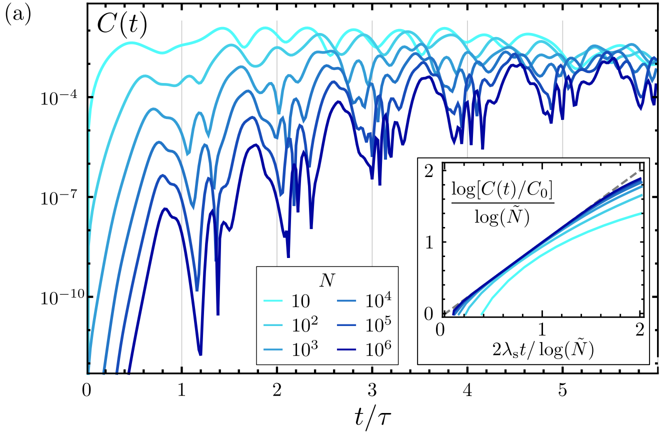
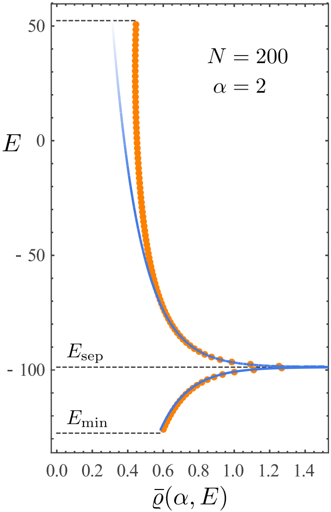



Coexistence of fast initial scrambling and long-time revivals
long-time not that long at all! (only logarithmic with $N$)

# The log time scale reconsidered

this we know...



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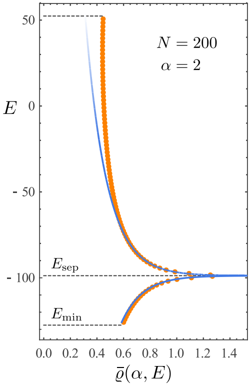

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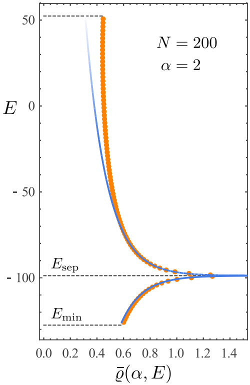
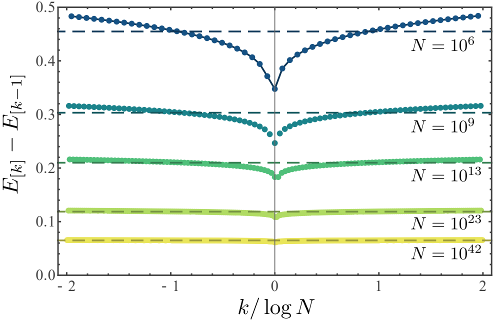



surprise!!

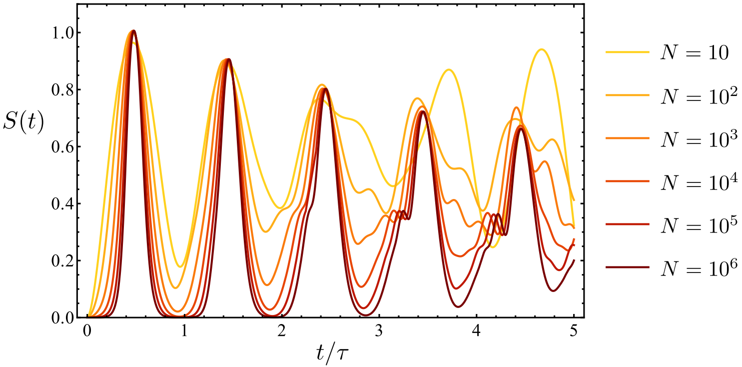



Despite the divergence, the spectrum is (locally) asymptotically homogeneous

$\rightarrow$ perfect coherent revivals at the log (not recurrence) time scale!!

# Robust?

Go from 3-site to 5-site (non integrable)



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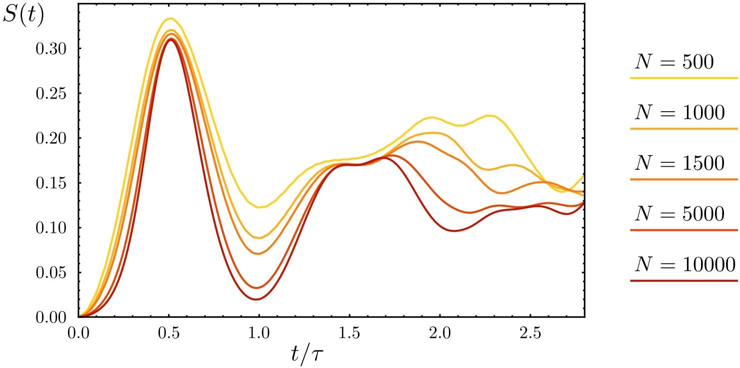


YES

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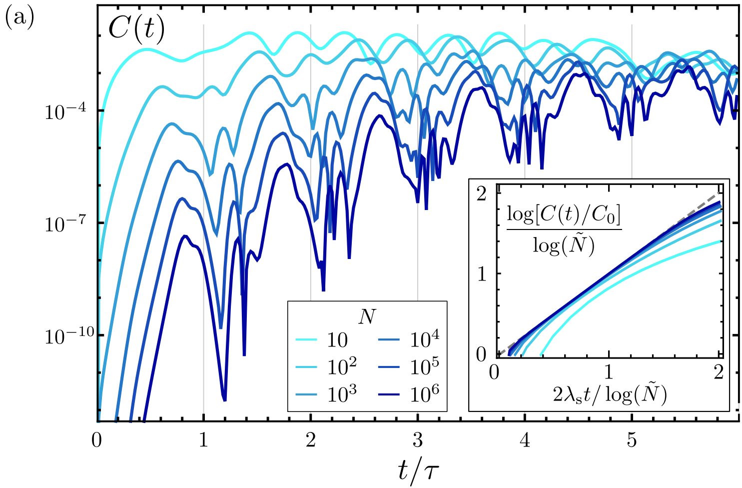

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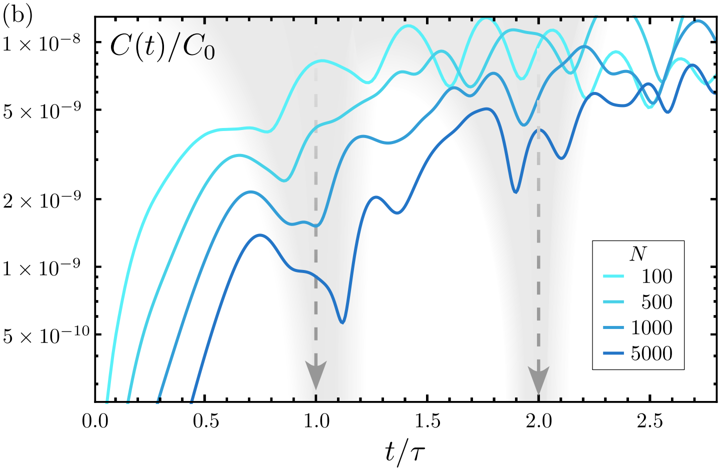


YES

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- (Truly) semiclassical methods a la Gutzwiller account for interference phenomena
- Please remember that! (semiclassics is NOT classics)
- Such ideas can be lifted to Fock space
- They account for fast scrambling and saturation of OTOCs in the chaotic case

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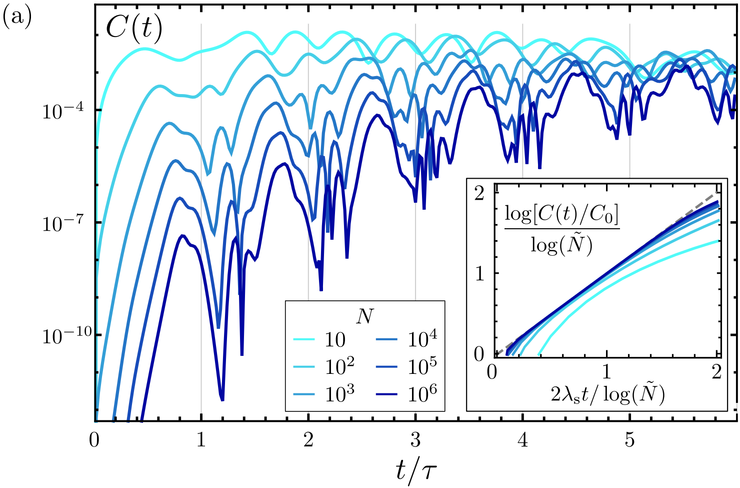
A unique SHORT time scale $\sim \log N$ ruling the initial scrambling, the breaking of quantum-classical correspondence and coherent revivals signaling re-entrant information




H. Hummel, B. Geiger, JDU, and K. Richter "Reversible quantum information spreading in many-body systems near criticallity"PRL **123**, 160401 (2019)