

# Excited-state quantum phase transitions: Phase space analysis

Qian Wang

Department of physics, Zhejiang Normal University, China  
CAMTP-Center for Applied Mathematics and Theoretical Physics, Slovenia

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# Outline

- ① Phase space distribution functions
- ② Husimi distribution function
- ③ Models
- ④ Results
- ⑤ Summary and outlook

## Phase space distribution functions

- Quantum mechanics:  $\hat{\rho}(\hat{p}, \hat{q})$ ,  $\langle \hat{A}(\hat{p}, \hat{q}) \rangle = \text{Tr}[\hat{\rho}(\hat{p}, \hat{q}) \hat{A}(\hat{p}, \hat{q})]$
- Phase space:  $F(p, q)$

$$\text{Tr}[\hat{\rho} \hat{A}(\hat{p}, \hat{q}) f(\xi, \eta)] = \int dp \int dq A(p, q) F^f(p, q)$$

- Distribution function for the density operator  $\hat{\rho}$

$$F^f(p, q) = \frac{1}{4\pi^2} \int d\xi \int d\eta \int dq' \langle q' + \frac{1}{2}\eta\hbar | \hat{\rho} | q' - \frac{1}{2}\eta\hbar \rangle \\ \times f(\xi, \eta) e^{i\xi(q' - q)} e^{-i\eta p}$$

- For an arbitrary operator  $\hat{A}(\hat{p}, \hat{q})$

$$A^f(p, q) = \frac{\hbar}{2\pi} \int d\xi \int d\eta \int dq' \langle q' + \frac{1}{2}\eta\hbar | \hat{A} | q' - \frac{1}{2}\eta\hbar \rangle \\ \times f(\xi, \eta) e^{i\xi(q' - q)} e^{-i\eta p}$$

# Phase space distribution functions

Types of the distribution functions (d.f.), corresponding choices of the function  $f$  and rules of association.

Distribution functions	Rules of association	$f$	References
Wigner d.f. ( $F^W$ )	Weyl ( $e^{i\xi q+i\eta p}$ $\leftrightarrow e^{i\xi \hat{q}+i\eta \hat{p}} = e^{z\hat{a}^\dagger-z^*\hat{a}}$ )	1	Wigner (1932,1971)
standard-ordered d.f. ( $F^S$ )	standard ( $e^{i\xi q+i\eta p}$ $\leftrightarrow e^{i\xi \hat{q}}e^{i\eta \hat{p}}$ )	$e^{-i\hbar\xi\eta/2}$ $= e^{(z^2-z^{*2})/4}$	Mehta (1964)
antistandard-ordered d.f. $(F^{AS})$ (Kirkwood d.f.)	antistandard ( $e^{i\xi q+i\eta p}$ $\leftrightarrow e^{i\eta \hat{p}}e^{i\xi \hat{q}}$ )	$e^{i\hbar\xi\eta/2}$ $= e^{-(z^2-z^{*2})/4}$	Kirkwood (1933) Rihaczek (1968)
normal-ordered d.f. ( $F^N$ ) (Glauber- Sudarshan $P$ function)	normal $(e^{i\xi q+i\eta p} = e^{z\alpha^*-z^*\alpha}$ $\leftrightarrow e^{z\hat{a}^\dagger}e^{-z^*\hat{a}})$	$e^{(\hbar\xi^2/4m\omega+\hbar m\omega\eta^2/4)}$ $= e^{ z ^2/2}$	Glauber (1963b,1965) Sudarshan (1963)
antinormal-ordered d.f. ( $F^{AN}$ ) ( $Q$ function)	antinormal $(e^{i\xi q+i\eta p} = e^{z\alpha^*-z^*\alpha}$ $\leftrightarrow e^{-z^*\hat{a}}e^{z\hat{a}^\dagger})$	$e^{(-\hbar\xi^2/4m\omega-\hbar m\omega\eta^2/4)}$ $= e^{- z ^2/2}$	Glauber (1965)
generalized antinormal-ordered d.f. $(F^H)$ (Husimi d.f.)	generalized antinormal $(e^{i\xi q+i\eta p} = e^{v\beta^*-v^*\beta}$ $\leftrightarrow e^{-v^*\hat{b}}e^{v\hat{b}^\dagger})$	$e^{(-\hbar\xi^2/4mk-\hbar mk\eta^2/4)}$ $= e^{- v ^2/2}$	Husimi (1940)

## Phase space distribution functions

- Bilinear in the wave function  $\psi$ ,  $f(\xi, \eta)$  to be independent of  $\psi$
- Real function
- Nonnegative
- Marginal distribution

$$\int dp F^f(p, q) = \langle q | \hat{\rho} | q \rangle, \quad \int dq F^f(p, q) = \langle p | \hat{\rho} | p \rangle$$

- Complete orthonormal condition

$$\int dq \int dp F_{nm}^f(p, q) F_{n'm'}^{f*}(p, q) = \frac{1}{2\pi\hbar} \delta_{nn'} \delta_{mm'}$$
$$\sum_{n,m} F_{nm}^f(p, q) F_{nm}^{f*}(p', q') = \frac{1}{2\pi\hbar} \delta(q - q') \delta(p - p')$$

$$F_{nm}^f(p, q) = \frac{1}{4\pi^2} \int d\xi \int d\eta \int dq' \phi_n^*(q' - \eta\hbar/2) \phi_m(q' + \eta\hbar/2)$$
$$\times f(\xi, \eta) e^{i\xi(q' - q)} e^{-i\eta p}$$

# Phase space distribution functions

Properties of the distribution functions (d.f.).

Physics Report **259** 147 (1995)

d.f.	Properties				
	(a) bilinear	(b) real	(c) nonnegative	(d) marginal distributions	(e) compete, orthonormal
Wigner d.f. ( $F^W$ )	yes	yes	no	yes	yes
standard-ordered d.f. ( $F^S$ )	yes	no	no	yes	yes
antistandard-ordered d.f. ( $F^{AS}$ )	yes	no	no	yes	yes
$P$ function ( $F^N$ )	yes	yes	no	no	no
$Q$ function ( $F^{AN}$ )	yes	yes	yes	no	no
Husimi d.f. ( $F^H$ )	yes	yes	yes	no	no

- Quantum optics
- Quantum chaos
- Condensed matter physics

## Husimi distribution function

- Husimi distribution function

$$Q(p, q) = \langle \zeta(p, q) | \hat{\rho} | \zeta(p, q) \rangle$$

- SU(2) spin- $j$  coherent states

$$|\zeta\rangle = (1 + |\zeta|^2)^{-j} e^{\zeta \hat{J}_+} |j, -j\rangle, \quad \frac{2j+1}{\pi} \int_{\mathbb{R}^2} |\zeta\rangle \langle \zeta| \frac{d^2\zeta}{(1 + |\zeta|^2)^2} = 1$$

- Parameterization of  $\zeta$

$$\zeta(p, q) = \frac{q - ip}{\sqrt{4 - (p^2 + q^2)}}, \quad \{p, q\} \in \Omega = \{(p, q) | p^2 + q^2 \leq 4\}$$

- Normalization relation of  $Q(p, q)$

$$\frac{2j+1}{4\pi} \int_{\Omega} Q(p, q) dp dq = 1$$

## Husimi distribution function

- Second moment

$$M_2 = \frac{2j+1}{4\pi} \int_{\Omega} Q^2(p, q) dp dq$$

- Wehrl entropy

$$W = -\frac{2j+1}{4\pi} \int_{\Omega} Q(p, q) \ln[Q(p, q)] dp dq$$

- Marginal distributions

$$Q(q) = \sqrt{\frac{2j+1}{4\pi}} \int Q(p, q) dp, \quad Q(p) = \sqrt{\frac{2j+1}{4\pi}} \int Q(p, q) dq$$

- Second moment and Wehrl entropy of  $Q(q)$  and  $Q(p)$

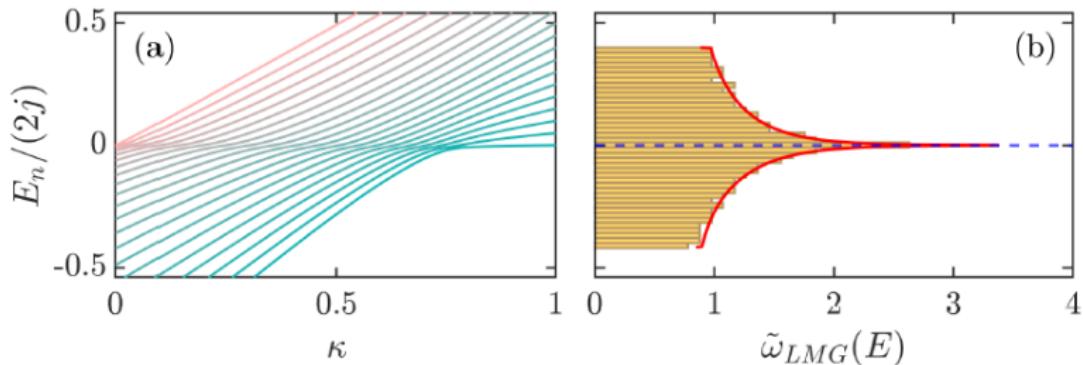
$$M_2^{(\mu)} = \sqrt{\frac{2j+1}{4\pi}} \int Q^2(\mu) d\mu,$$

$$W^{(\mu)} = -\sqrt{\frac{(2j+1)}{4\pi}} \int Q(\mu) \ln[Q(\mu)] d\mu$$

# Models–Lipkin-Meshkov-Glick (LMG) model

$$\hat{H}_{LMG} = -\frac{4(1-\kappa)}{N} \hat{j}_x^2 + \kappa \left( \hat{j}_z + \frac{N}{2} \right)$$

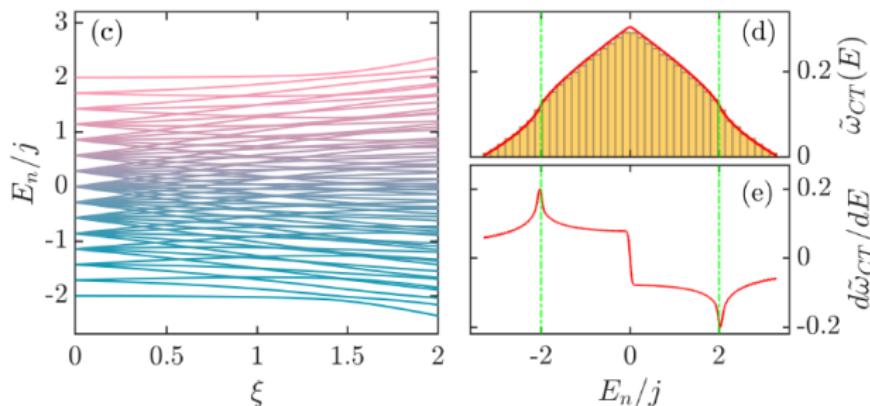
- $[\hat{H}_{LMG}, \hat{\mathbf{j}}^2] = 0, [\hat{H}_{LMG}, \hat{\Pi}] = 0, \Pi = e^{i\pi(j+m)}$
- Ground state QPT:  $\kappa_c = 4/5$ , paramagnetic phase ( $\kappa < \kappa_c$ ) vs. ferromagnetic phase ( $\kappa > \kappa_c$ )
- ESQPT at  $E_c = 0$
- Density of states:  $\omega_{LMG}(E) = \sum_n \delta(E - E_n)$



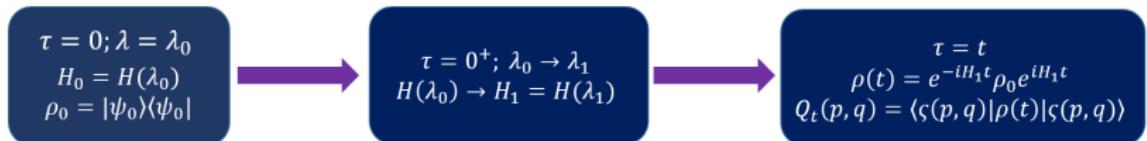
## Models–Coupled top (CT) model

$$\hat{H}_{CT} = \hat{J}_{1z} + \hat{J}_{2z} + \frac{\xi}{j} \hat{J}_{1x} \hat{J}_{2x}$$

- $\hat{J}_k^2 |j, m_k\rangle = j(j+1) |j, m_k\rangle$ ,  $[\hat{H}_{CT}, \hat{J}_1] = [\hat{H}_{CT}, \hat{J}_2] = 0$
- $[\hat{H}_{CT}, \mathcal{P}] = 0$ ,  $[\hat{H}_{CT}, \Pi] = 0$ ,  $\Pi = e^{i\pi(2j+m_1+m_2)}$
- Ground state QPT: Ferromagnetic phase ( $\xi < \xi_c$ ) → Paramagnetic phase ( $\xi > \xi_c$ ) at  $\xi_c = 1$
- ESQPT:  $E_c/j = \pm 2$ ,  $\omega_{CT} = \sum_n \delta(E - E_n)$



## Results–Initial Setting



- ① Initial state:  $\rho_0 = |\psi_0\rangle\langle\psi_0| = |GS\rangle\langle GS|$
- ② Quantum protocol
  - LMG model:  $\hat{H}_{LMG} \rightarrow \hat{H}_{LMG} + \eta(\hat{J}_z + N/2)$ ,  $\eta_c = 2 - 5\kappa/2$
  - CT model:  $\hat{H}_{CT}(\xi_0) \rightarrow \hat{H}_{CT}(\xi_1)$ ,  $\xi_c = 2\xi_0/(\xi_0 + 1)$
- ③ Time evolving:  $\rho(t) = e^{-iH^f t} \rho_0 e^{iH^f t}$

$$Q_t(p, q) = \langle \zeta(p, q) | \rho(t) | \zeta(p, q) \rangle$$

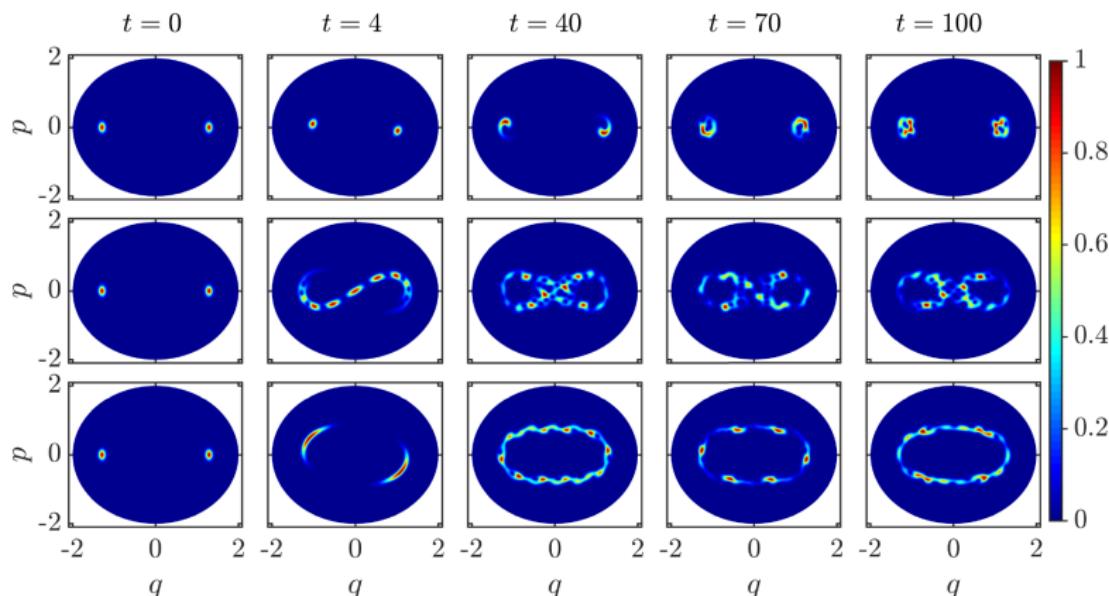
- ④ Long-time averaged state

$$\bar{\rho} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho(t) dt, \quad \overline{Q}(p, q) = \langle \zeta(p, q) | \bar{\rho} | \zeta(p, q) \rangle$$

# Results in LMG model

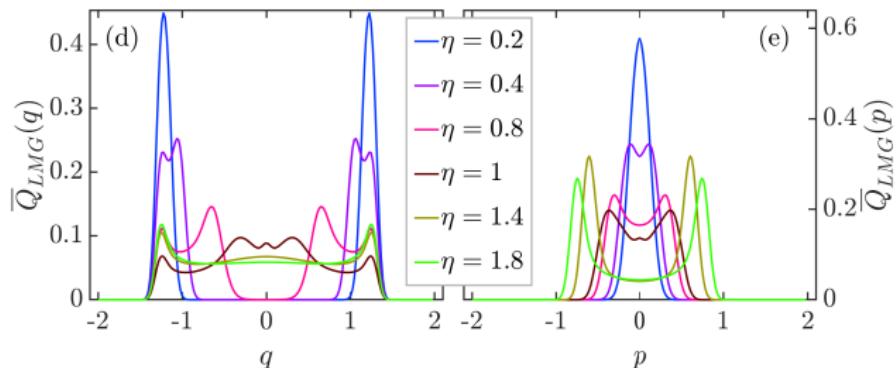
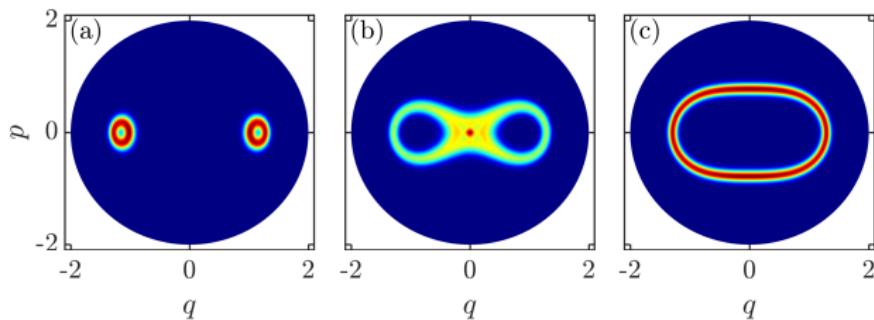
Husimi distribution function of  $\rho(t)$

$$Q_t(p, q) = \left| \sum_n e^{-iE_n t} \langle \zeta(p, q) | E_n \rangle \langle E_n | \psi_0 \rangle \right|^2$$



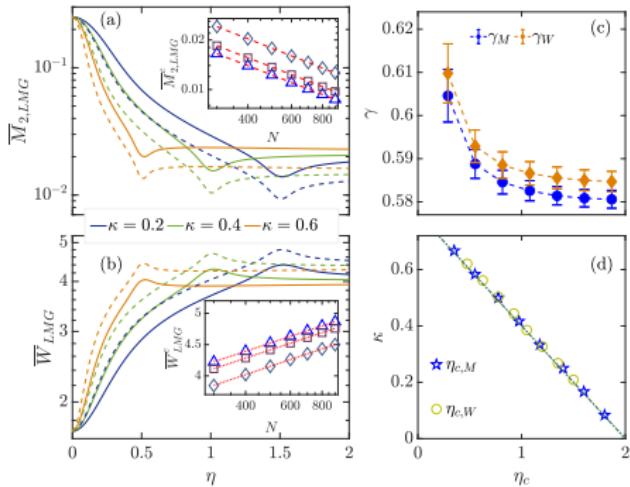
# Husimi distribution function of $\bar{\rho}$

$$\overline{Q}_{LMG}(p, q) = \sum_n |\langle \zeta(p, q) | E_n \rangle|^2 |\langle E_n | \psi_0 \rangle|^2$$



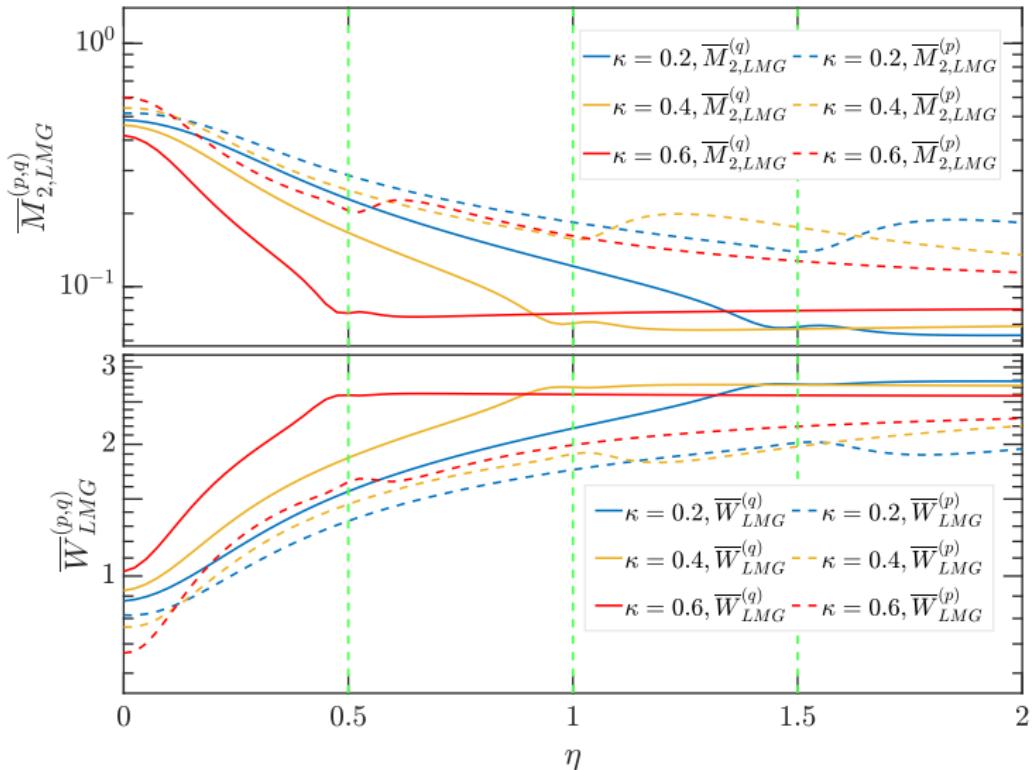
$$\overline{M}_{2,LMG} = \frac{2j+1}{4\pi} \int_{\Omega} \overline{Q}^2(p, q) dp dq$$

$$\overline{W}_{LMG} = -\frac{2j+1}{4\pi} \int_{\Omega} \overline{Q}(p, q) \ln[\overline{Q}(p, q)] dp dq$$



- $\bar{\rho}$  has the maximal extension at the critical point
- $\overline{M}_{2,LMG}^c \sim N^{-\gamma_M}$ ,  $\overline{W}_{LMG}^c \sim \gamma_W \ln(N)$
- $\overline{M}_{2,LMG}^m$  and  $\overline{W}_{LMG}^m$  provide a reliable estimation of the critical point

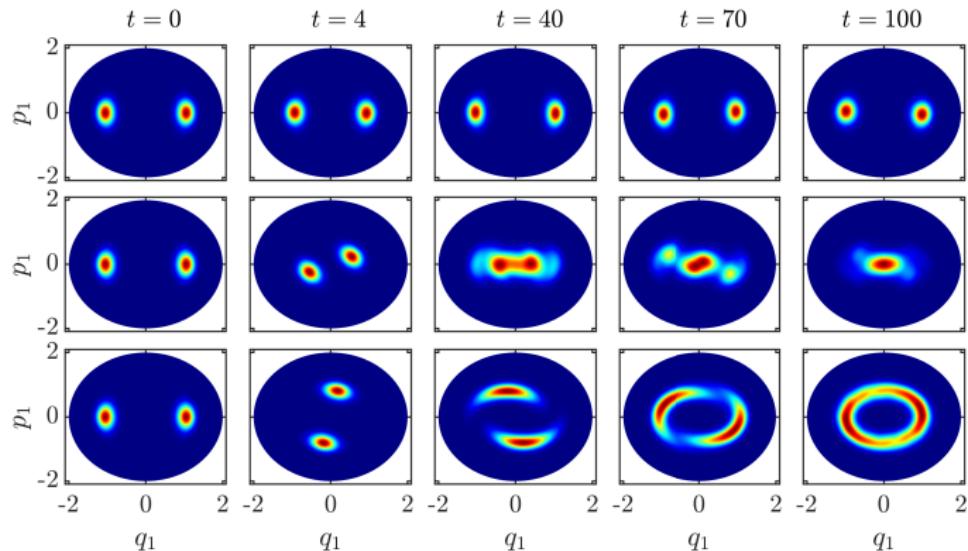
- Behaviors of  $\overline{M}_{2,LMG}^{(p,q)}$  and  $\overline{W}_{LMG}^{(p,q)}$



# Results in CT model

Projected Husimi distribution function of  $\rho_t$

$$Q_t(p, q) = \langle \zeta(p_1, q_1) | \rho_1^t(\xi_1) | \zeta(p_1, q_1) \rangle, \quad \rho_1^t = \text{Tr}_2[\rho_t(\xi_1)]$$



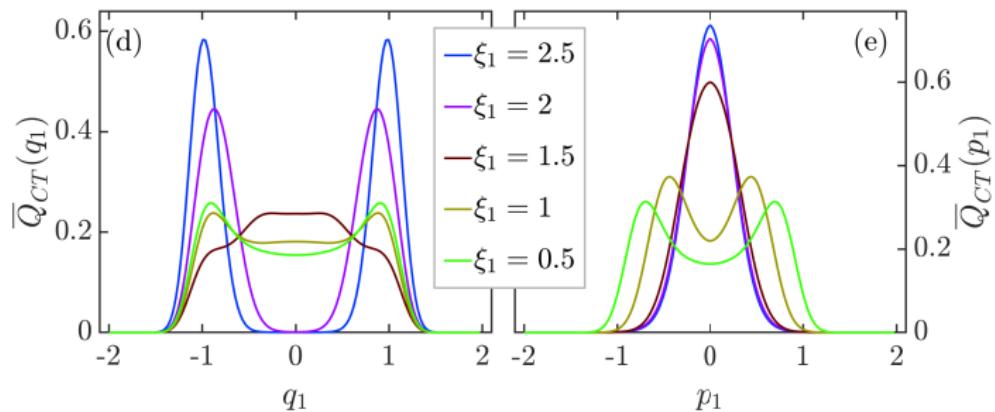
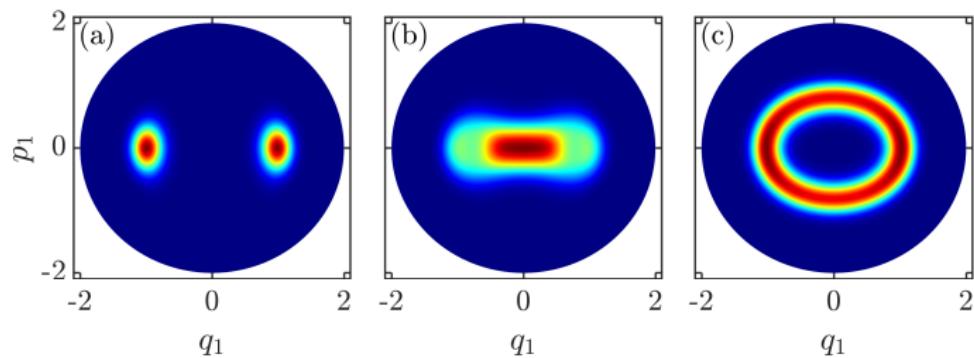
- Long-time averaged reduced density matrix

$$\bar{\rho}_1(\xi_1) = \text{Tr}_2[\bar{\rho}(\xi_1)], \quad \bar{\rho}(\xi_1) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \rho_t(\xi_1) dt$$

- Long-time averaged projected Husimi function

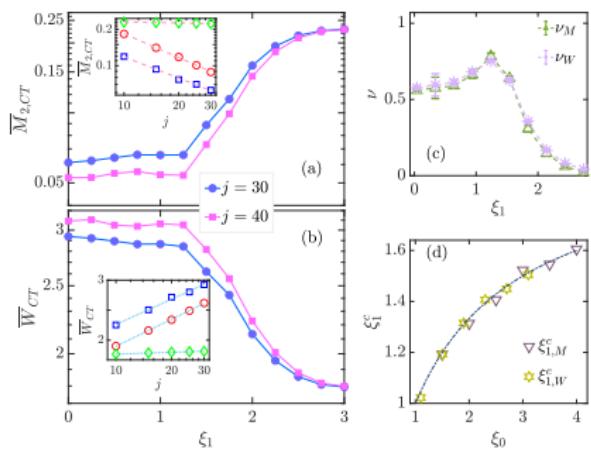
$$\begin{aligned}\overline{Q}_{CT}(p_1, q_1) &= \langle \zeta(p_1, q_1) | \bar{\rho}_1(\xi_1) | \zeta(p_1, q_1) \rangle \\ &= \sum_n |\langle \Psi_0 | E_n \rangle|^2 \langle \zeta(p_1, q_1) | \rho_1^{(n)}(\xi_1) | \zeta(p_1, q_1) \rangle\end{aligned}$$

$$\rho_1^{(n)}(\xi_1) = \text{Tr}_2[|E_n(\xi_1)\rangle\langle E_n(\xi_1)|]$$



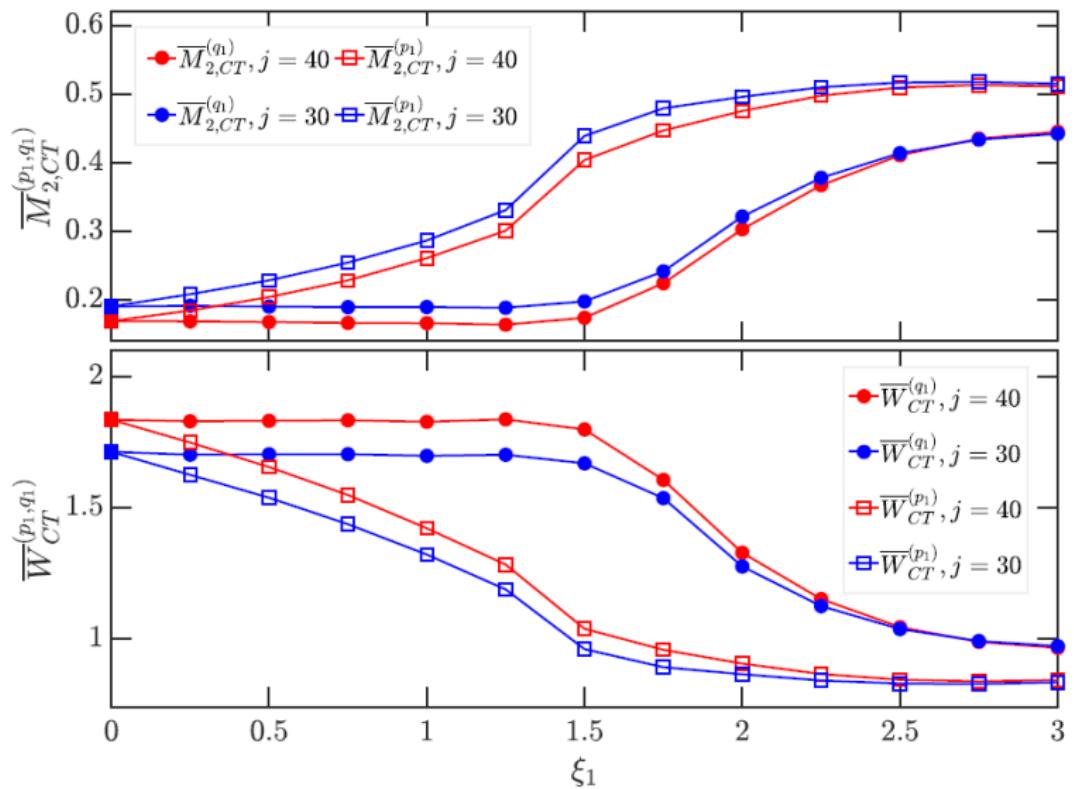
$$\overline{M}_{2,CT} = \frac{2j+1}{4\pi} \int_{\Omega} \overline{Q}_{CT}(p_1, q_1) dp_1 dq_1$$

$$\overline{W}_{CT} = -\frac{2j+1}{4\pi} \int_{\Omega} \overline{Q}_{CT}(p_1, q_1) \ln[\overline{Q}_{CT}(p_1, q_1)] dp_1 dq_1$$



- The quantum state shows different degrees of localization
- $\overline{M}_{2,CT} \sim j^{-\nu_M}$ ,  $\overline{W}_{CT} \sim \nu_W \ln(j)$
- The critical point can be estimated by  $d\nu_{M(W)}/d\xi|_{min}$

- Behaviors of  $\overline{M}_{2,CT}^{(p,q)}$  and  $\overline{W}_{CT}^{(p,q)}$



# Summary and outlook

- Summary
  - ① Husimi distribution function exhibits distinct dynamical behaviors in different phases of the ESQPT
  - ② The Husimi function of long-time averaged state shows different properties as the system crossovers the critical point
  - ③ The scaling behaviors of  $\overline{M}_2$  and  $\overline{W}$  reveal the characteristics of the ESQPT
- Outlook
  - ① The relation between the features of the Husimi distribution function and the first-order ESQPT (Dicke model)
  - ② Phase space multifractal features of the quantum states and ESQPT

## **Out-of-time-order correlator and ESQPT**

# Out-of-time-order correlator (OTOC)

First introduced by Larkin and Ovchinnikov in 1969 and rekindled from 2015

SOVIET PHYSICS JETP

VOLUME 28, NUMBER 6

JUNE, 1969

QUASICLASSICAL METHOD IN THE THEORY OF SUPERCONDUCTIVITY

A. I. LARKIN and Yu. N. OVCHINNIKOV

Institute of Theoretical Physics, USSR Academy of Sciences

Submitted June 6, 1968

Zh. Eksp. Teor. Fiz. 55, 2262-2272 (December, 1968)

- A. Kitaev, *A simple model of quantum holography* KITP program (2015).

$$C(t) = -\langle [W(t), V(0)]^2 \rangle = F_d - \left\langle V(0)^\dagger W(t)^\dagger V(0) W(t) + W(t)^\dagger V(0)^\dagger W(t) V(0) \right\rangle$$

Scrambling time  $t_s$ :  
Initial states become indistinguishable       $F(t) \sim F_d - \epsilon e^{\lambda t}$

Dissipation time  $t_d$ :  
Local observables relax to equilibrium  
constant

Scaling with Lyapunov exponent

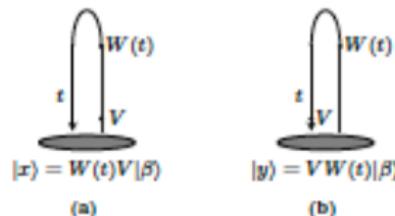
OTOC: Measures overlap between applying  $V$  then  $W$  after time  $t$ , and applying  $W$  at  $t$ , reversing time, then  $V$

[J. Maldacena, et al., JHEP 2016, 106 (2016)]

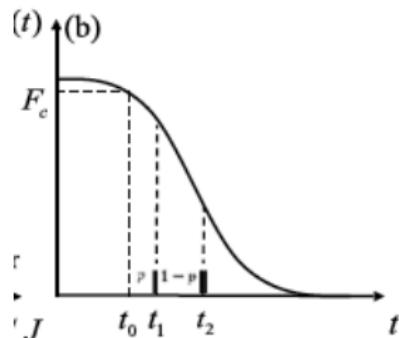
# OTOC

$$F(t) = \langle W^\dagger(t)V^\dagger(0)W(t)V(0)\rangle$$

- $F(t) = \langle x|y\rangle$



- $\text{Re}[F(t)] = \alpha_0 - \epsilon e^{\lambda_L(t-|x|/\nu_B)}$



- $W(t) \rightarrow x(t)$ ,  $V(0) \rightarrow p(0)$ , in the classical limit

$$C(t) = -\langle [W(t), V(0)]^2 \rangle \rightarrow \langle \{x(t), p(0)\}_{\text{PB}}^2 \rangle$$

$$\sim \left\langle \left( \frac{\partial x(t)}{\partial x_0} \right)^2 \right\rangle \sim e^{2\lambda_L^c t}$$

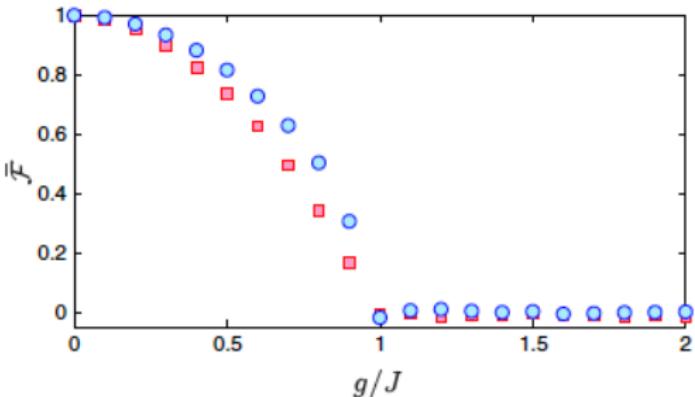
$\lambda_L$  is the quantum counterpart of  $\lambda_L^c$ .

- The upper bound

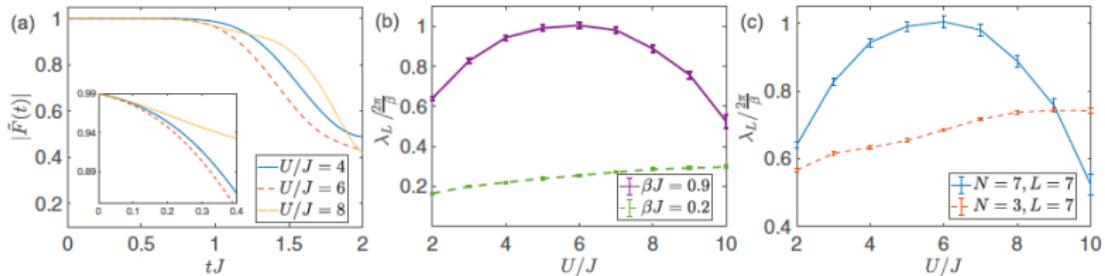
$$\lambda_L \leq \frac{2\pi}{\beta}$$

- The OTOC has been measured experimentally with in ion traps and nuclear magnetic resonance platforms.

- Probing the phase transitions in quantum many-body systems



[PRL 121, 016801 (2018)]

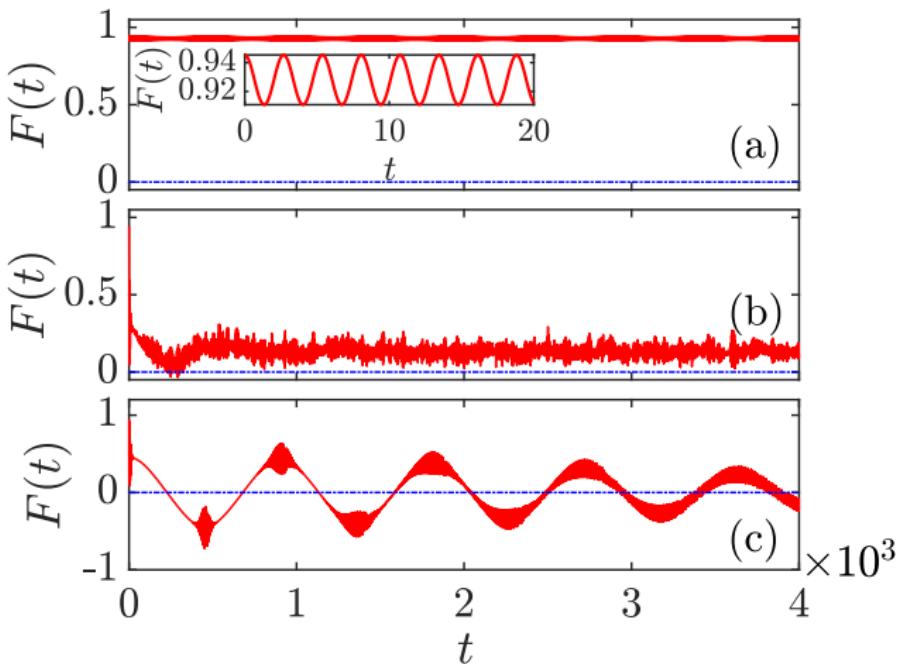


[PRB 96, 054503 (2017)]

# Time evolution of OTOC

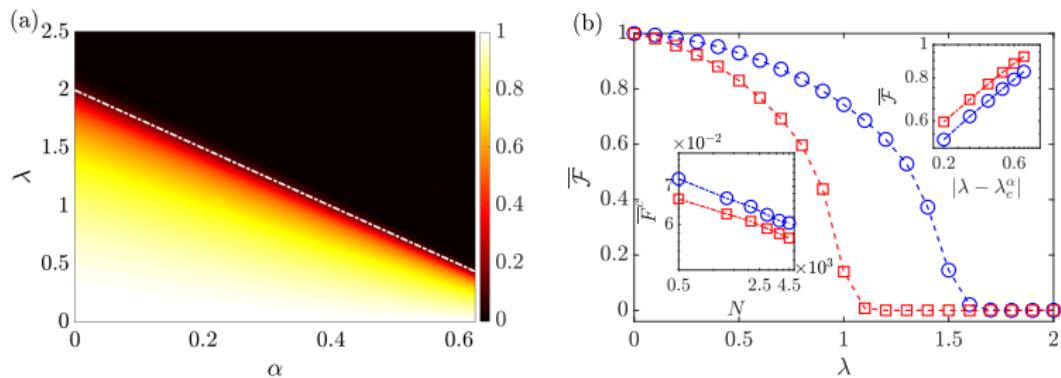
$$W(0) = V(0) = S_x/\mathcal{S}, W(t) = e^{iH(\lambda)t} S_x e^{-iH(\lambda)t}/\mathcal{S}$$

$$F(t) = \frac{1}{\mathcal{S}^4} \langle \psi_0^\alpha | e^{iH(\lambda)t} S_x e^{-iH(\lambda)t} S_x e^{iH(\lambda)t} S_x e^{-iH(\lambda)t} S_x | \psi_0^\alpha \rangle$$



# Long-time averaged OTOC

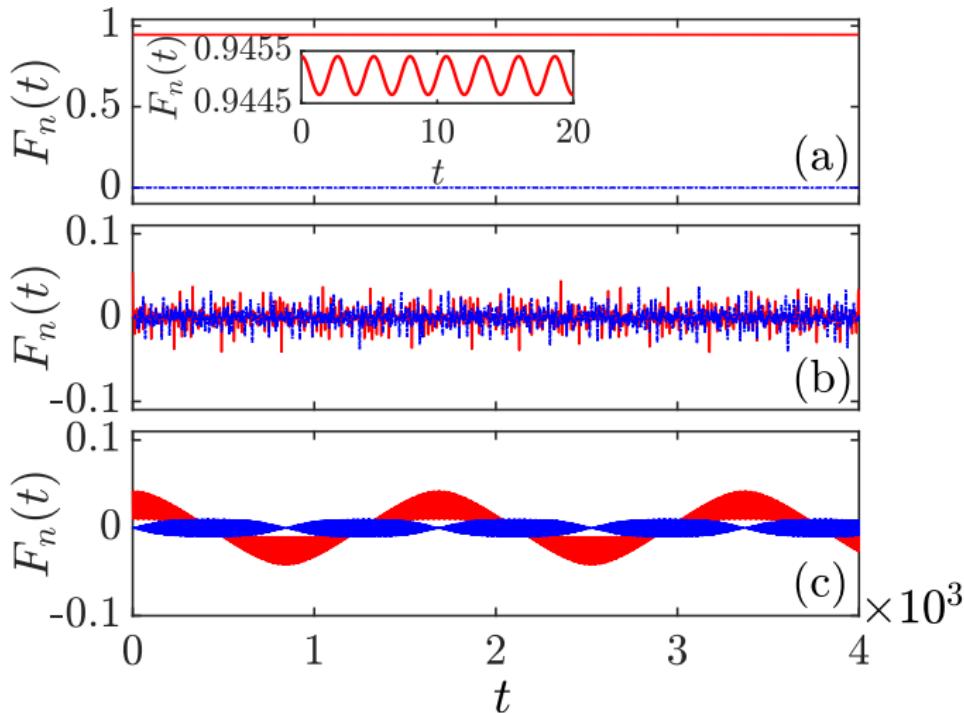
$$\overline{F} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_R(t) dt, \quad \overline{\mathcal{F}} = \frac{\overline{F}}{\overline{F}(\lambda = 0)}$$



- $\overline{F}^c = \overline{F}(\lambda = \lambda_c) \propto N^{-\mu}$  with  $\mu \approx 0.084(8)$
- $\overline{F} \propto |\lambda - \lambda_c^\alpha|^{\gamma_\lambda}$  with  $\gamma_\lambda \approx 0.36(1)$

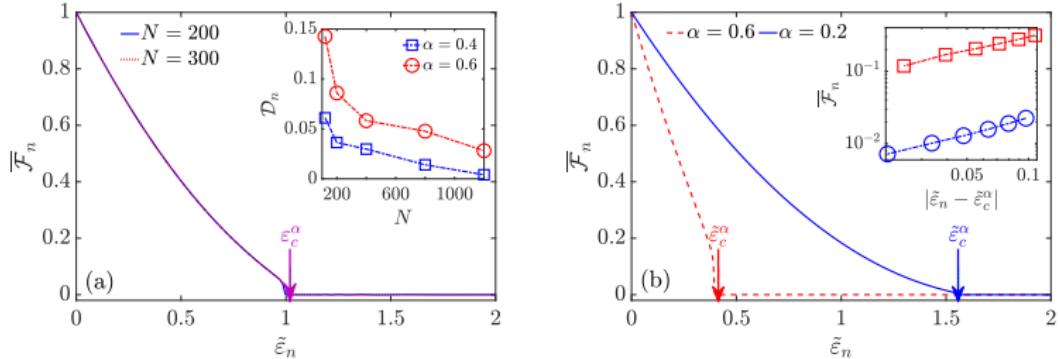
# Micro-canonical OTOC (MOTOC)

$$F_n(t) = \frac{1}{S^4} \langle n | e^{iH_0^\alpha t} S_x e^{-iH_0^\alpha t} S_x e^{iH_0^\alpha t} S_x e^{-iH_0^\alpha t} S_x | n \rangle$$



# Long-time averaged MOTOC

$$\overline{F}_n = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_n(t) dt, \quad \overline{\mathcal{F}}_n = \frac{\overline{F}_n}{\overline{F}_0}$$



- $\tilde{\varepsilon}_n = 2(E_n - E_0)/(E_{max} - E_0)$ ,  $\tilde{\varepsilon}_c^\alpha = -2E_0/(E_{max} - E_0)$
- $\mathcal{D}_n$  is the increase and/or decrease amplitude of  $\overline{\mathcal{F}}_n$  around the critical point
- $\overline{\mathcal{F}}_n \propto |\tilde{\varepsilon}_n - \tilde{\varepsilon}_c^\alpha|^{\gamma_\varepsilon}$  with  $\gamma_\varepsilon \approx 0.69(5)$

## Summary

- ESQPT leaves significant imprints on the OTOC dynamics
- The presence of an ESQPT and its critical point, as well as the different phases can be detected via OTOC
- Long-time averaged OTOC behaves as the order parameter of ESQPT

Open questions:

- OTOC dynamics in other kinds of ESQPTs, such as the ESQPT in Dicke model
- The requirements that  $W$  and  $V$  should fulfill in order to make sure the OTOC performed as a detector between different phases in quantum many-body systems

**Thanks & Questions**