Approaching to Minkowski space in the framework of DSEs

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Outline

1. **Background**
2. **On shell information of hadron**
   - First-principle treatment of the truncation for heavy-light mesons
   - Meson Masses and decay constants
3. **Analytical continuation of propagator**
4. **Summary**

Dyson-Schwinger Equations

Dyson-Schwinger equations approach is a powerful nonperturbative continuum method to study QCD.

DSEs are the equations of motion for n-point Green function in quantum field theory derived from the variation of action:

$$\left\langle \frac{\delta S[\phi]}{\delta \phi(x)} \right\rangle_J = J(x)$$

DSEs

The DS equation for quark propagator could be expressed as

$$S(\not{p})^{-1} = i\not{p} + m + \Sigma(\not{p}) = iA(p^2)\not{p} + B(p^2)\,, \qquad (1)$$

where

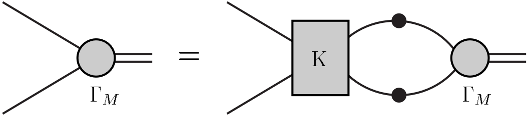
$$\Sigma(\not{p}) = \frac{16}{3}\int \frac{d^4q}{(2\pi)^4}\, g^2 D_{\mu\nu}(\not{p}-\not{q})\gamma_\mu S(\not{q})\Gamma_\nu(\not{q},\not{p})$$

Bethe-Salpeter equations (BSEs)

The homogeneous BSEs are eigenvalue problems for meson bound states which could be described as following.

$$\left[\Gamma_M^{ab}(p;P)\right]_{tu} = \int_q^{\Lambda} K_{tu}^{rs}(p,q;P)\,\left[\chi_M^{ab}(q;P)\right]_{sr}\,, \tag{2}$$

where
$\chi_M^{ab}(q;P) := S^a(q_+)\Gamma_M^{ab}(q;P)S^b(q_-)$ with $q_+ := q + \eta\, P$,
$q_- := q - \bar{\eta}\, P$



Practical challenge

Practical challenge in studying QCD:

The solution of DSEs is obtained in Euclidean space:

- That is where all the results of perturbation theory, the renormalisation group and lattice-QCD are known
- This enables reliable constraints on input and checks on output

One needs to return to the Minkowski space:

- The mass pole of hadron lies on real axis.
- The spectral function of propagators is related to its imaginary part on real axis.

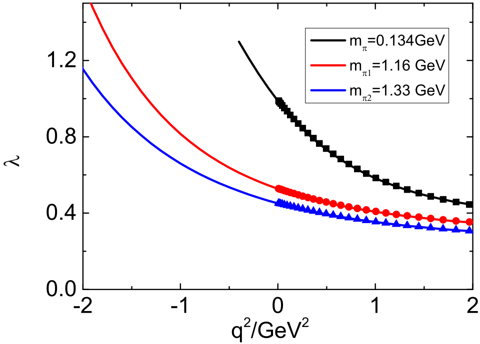
Extrapolation scheme for hadron properties

When computing the meson properties, people need the information in complex plane. However, the singularities in the complex plane make obstacles especially when computing the heavy-light mesons.

- Computing the eigenvalues $\lambda(P^2)$of BS equations in Euclidean space
- Extrapolating into complex plane and finding the location of P^2 where $\lambda(P^2 = -M^2) = 1$
- Extrapolating the decay constant till $P^2 = -M^2$

Ground states and excited states of pion

First, we try this framework for pion:



- The mass of ground state is the same as the direct computation in complex plane.
- The masses of excited states and the exotic states can be obtained.
- The extrapolated decay constant is $f_\pi = 0.098$ GeV.

Heavy-light meson

The decay constants of heavy-light mesons play an important role in the phenomenological description of various processes relevant for heavy-flavour physics. For example, it will provide useful description of semileptonic form factors and also of non-leptonic decay rates.

- These decay constants have been widely measured by many experimental collaborations, for example, Belle, BABAR and CLEO collaboration.
- The theoretical computation for these decay constants is of great importance for determining the CKM matrix elements and verifying standard model.

Model and truncation for heavy light mesons

Rainbow-Ladder truncation:

- Qin-Chang model[1]

$$\mathcal{G}_{IR}(k^2) = \frac{8\pi^2}{\omega^4} D e^{-k^2/\omega^2} + \frac{4\pi}{k^2}\alpha_{pQCD}(k^2). \tag{3}$$

- self-consistent truncation

$$\gamma_\mu S(k)\gamma_\nu \to \gamma_\mu S(k+P/2)\Gamma_M(k,P)S(k-P/2)\gamma_\nu. \tag{4}$$

[1] Si-xue Qin, Lei Chang, Yu-xin Liu, Craig D. Roberts, and David J. Wilson, Phys. Rev. C 84, 042202

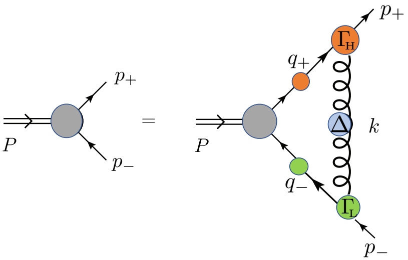
Beyond rainbow ladder approximation

The RL approximation is good for a meson composed of two equal quarks, since the kernel used in both gap equations is common and then a self-consistent kernel in BSE can be defined.
If to consider a meson consisting of two quarks with different masses,

- the interaction kernels (K_{LL} and K_{HH}, respectively) appearing in the two gap equations will be different;
- the kernel appearing in the BSE describing the $Q_H Q_L$ meson, to be denoted by K_{HL}, is unique, and a-priori is neither K_{LL} nor K_{HH}.

Therefore, the rainbow-ladder approximation is not suitable for heavy-light mesons.

General considerations



Considering the kernel of heavy-light mesons, the qualitative effect of including two different vertices is that the combined interaction kernel

$$K_{HL} = \Gamma^{\mathrm{H}}_{\mu}(k, p_+, q_+)\, \mathbf{G}(k^2)\, P^{\mu\nu}(k)\, \Gamma^{\mathrm{L}}_{\nu}(k, p_-, q_-) \qquad (5)$$

The improved interaction kernel

In order to model the dependence of Γ_μ on the quark mass we will use the simplest possible nontrivial approximation, namely the so called “central” (C) Ball-Chiu Ansatz, which is simply

$$\Gamma^{f}_{\mathbf{C}\,\mu}(k,q,p) = \frac{1}{2}\left[A_f(q) + A_f(p)\right]\gamma_\mu \tag{6}$$

Evidently, $\Gamma^{\mathrm{H}}_{\mu}(k,p_+,q_+)$ and $\Gamma^{\mathrm{L}}_{\nu}(k,p_-,q_-)$ depend on three kinematic variables each. To simplify the analysis, we will evaluate them at the symmetric point, namely $p_+^2 = q_-^2 = k^2 = p_-^2 = q_-^2$, so that the K_{HL} becomes a function of k only, namely

$$K_{HL}(k) \propto \gamma_\mu \underbrace{A_{\mathrm{H}}(k^2)\,\mathbf{G}(k^2)\,A_{\mathrm{L}}(k^2)}_{\mathcal{I}(k^2)} P^{\mu\nu}(k)\gamma_\nu \tag{7}$$

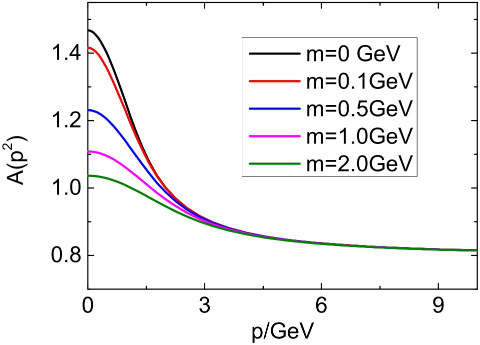
The improved interaction kernel

To correctly introduce in this correction, we start from the chiral limit with RL approximation:

- Even though the parameters people choose in chiral limit leads to good results for pion, kaon and ρ meson, the parameterized interaction strength is actually much larger than its physical value.
- In the chiral limit, $\mathbf{G}(k^2) = \mathcal{I}_{00}(k^2)$ is approximately $A_0^2(k^2)\mathcal{I}^{phys}(k^2)$ owing to WTI, where $\mathcal{I}^{phys}(k^2)$ is the kernel with the physical interaction strength.
- In the heavy-light system, the interaction kernel for heavy-light system is $\mathcal{I}_{HL}(k^2) = A_{\rm H}(k^2)\,\mathcal{I}^{phys}(k^2)\,A_{\rm L}(k^2) = A_{\rm H}(k^2)\,\mathbf{G}(k^2)\,A_{\rm L}(k^2)/A_0^2(k^2)$.

Numerical results for quark propagators

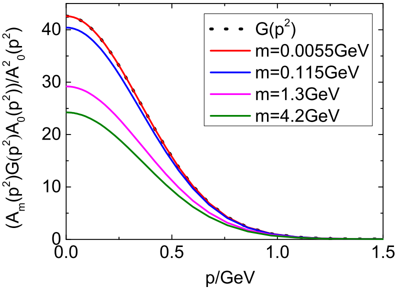
The solutions obtained for various current quark masses:



- As the mass increases the corresponding $A(p^2)$ becomes flatter, turning practically into a constant.

The improved interaction kernel

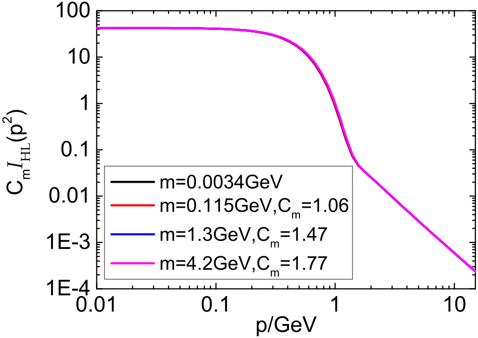
The chiral-limit curve is the highest, which represents the fact that the interaction strength of a heavy-light system is typically smaller than that of a light system.



- The large current quark mass naturally offers a large energy scale into the momentum transferred, the quark gluon vertex will tend to act as the bare one.
- Therefore, the enhanced correction of the vertex in the light systems will get gradually reduced, and will finally collapse to the bare vertex in the heavy system.

Numerical results of kernel

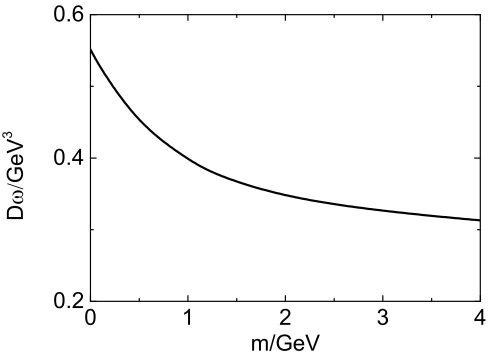
This improved interaction kernel can be easily described by adding a common factor on the bare vertex:



- Common factor:$C_m = A_m(0)/A_0(0)$
- Equivalently, it can be described by varying the interaction parameter $D\omega$ as a function of quark mass
- An improved truncation beyond Rainbow-ladder approximation.

Effective coupling strength

The effective coupling strength decreases as the current quark mass increases:



Pseudoscalar mesons

We then compute the mass and decay constant for pseudoscalar mesons:

m_{PS}/GeV	D	D_s	B	B_s	B_c
Expt.	1.87	1.97	5.28	5.37	6.28
Here	1.82	1.93	5.01	5.04	6.19
RL	1.85	1.97	5.27	5.38	6.36
lQCD	1.87	1.97	5.28	5.37	/
f_{PS}/GeV	D	D_s	B	B_s	B_c
Expt.	0.146	0.182/0.187	0.149	/	/
Here	0.159	0.179	0.146	0.158	0.406
RL	0.109	0.139	0.074	0.102	0.148
lQCD	0.147	0.175	0.137	0.162	/

- f_D=0.159 GeV, f_{D_s}=0.179 GeV, $f_{D_s}/f_D = 1.14$
- Decay constant of b-light meson is smaller than that of respective c-light meson.

Vector mesons

We can also do the same thing to vector mesons:

m_V/GeV	$D^\star$	$D_s^\star$	$B^\star$	$B_s^\star$	$B_c^\star$
Expt.	2.01	2.11	5.33	5.42	/
Here	1.89	1.93	5.35	5.55	6.20
RL	2.04	2.17	5.32	5.42	6.44
lQCD	2.01	2.11	5.32	5.41	/
f_V/GeV	$D^\star$	$D_s^\star$	$B^\star$	$B_s^\star$	$B_c^\star$
Expt.	/	/	/	/	/
Here	0.179	0.185	0.123	0.156	0.391
RL	0.113	0.127	0.129	0.141	0.127
lQCD	0.158	0.190	0.131	0.158	/

- For D meson, the decay constant of vector meson is larger than that of the respective pseudoscalar meson.
- For B meson, the decay constant of pseudoscalar meson is larger.

Spectral representation of propagator

The Kallen-Lehmann spectral representation gives a general expression for the two point Green function of quantum field theory, which is written as:

$$\Delta(p^2) = \frac{1}{2\pi} \int_{-\infty}^{0} d\mu^2 \rho(\mu^2) \frac{1}{-p^2 + \mu^2}, \tag{8}$$

with $\rho(p^2) = 2Im\Delta(p^2)$.

This dispersion relation includes a condition that the singularities of the propagator are located only on the real axis.

Singularities of propagators

- In the scalar or Abelian gauge field theory, the singularity structure is clear. All the singularities are located on the real axis, and then yield a positive-definite spectral function.
- However, in QCD, the situation is more complicated which has been already indicated by the confinement phenomenon.

Singularities of QCD

After giving up the positivity-definite condition, and forcing the integral spectral representation, people will find a spectral function with negative part for quark and gluon propagators.

The non positive-definite spectral function offers a stronger damping effect in the behaviour of propagator, which indicates that:

- Either there're higher order singularities on real axis than the simple mass poles and branch cuts,
- Or there are complex singularity structures on the upper plane of the p^2-complex plane.

Singularities of QCD

If there're complex singularities on the upper plane, there is actually contradiction among the definitions.

This extension of the spectral representation can only be employed after giving up the relation between the spectral function and the imaginary part of the respective function, that is,

$\rho(p^2) = 2Im\Delta(p^2)$, *has been broken in this case.*

Singularities of QCD

This can be easily shown via a simple model of propagator with one pair of complex conjugate poles:

$$\Delta(p^2) = \frac{1}{p^2 + z^2} + \frac{1}{p^2 + z^{*2}}$$

.

- The direct continuation of this propagator into Minkowski space would be $\frac{1}{-p^2+z^2} + \frac{1}{-p^2+z^{*2}}$, and it is easy to know that the imaginary part of this function on real axis with $\epsilon \to 0$ tends to vanish, and $\rho(p^2) = 2Im\Delta(p^2 - i\epsilon) \to 0$
- The spectral representation $\Delta(-p^2) = \frac{1}{2\pi}\int_{-\infty}^{0} d\mu^2 \rho(\mu^2)\frac{1}{p^2+\mu^2}$ means that the spectral function cannot be zero.

Singularities of QCD

The integral spectral representation is not compatible with the definition with imaginary part of propagator.

This contradiction reflects that

- Since there're singularities on the upper plane, the direct continuation and the continuation with spectral representation is not equivalent any more.
- The oscillated spectral representation can still reconstruct the function in Euclidean space, but it is no longer the analytical continuation of the original function.

Singularities of QCD

This offers the chance to approach the analyticity of propagators. People can employ the two different continuation methods and check their compatibility.

- If the results from two continuation methods coincide, the singularity structure on the complex plane will be directly excluded.
- Otherwise, there will be complex singularity structures on the upper plane.

Practical use of Hilbert transform

Practically, since what we can know from DSEs, is the Euclidean data of propagator, to know the spectral function in the integral becomes an ill-posed problem:

$$\Delta(p^2) = \frac{1}{2\pi}\int_{-\infty}^{0} d\mu^2 \rho(\mu^2)\frac{1}{-p^2+\mu^2}, \tag{9}$$

However, the spectral representation can be generalized by the Hilbert transform straightforwardly, which are:

$$Ref(p^2) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\mu^2 \frac{Imf(\mu^2)}{-p^2+\mu^2}, \tag{10}$$

$$Imf(p^2) = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\mu^2 \frac{Ref(\mu^2)}{-p^2+\mu^2}. \tag{11}$$

Hilbert transform

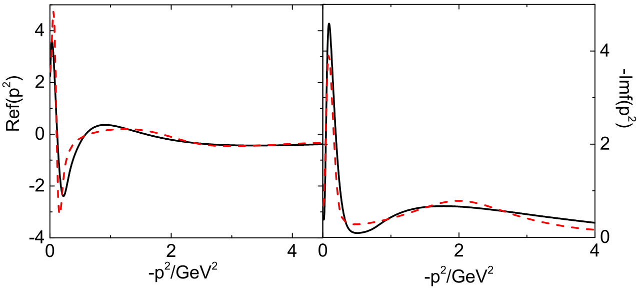
- If the singularities of function $f(p^2)$ are located only on the real axis, the upper panel of Hilbert transform will return to the spectral representation.
- As long there're no singularities on the upper complex plane of p^2 and $f(p^2 \to \infty) \to 0$, the relation can hold.

Owing to the real analyticity of propagators, that is, $F(z) = F^*(z^*)$, the lower plane singularities will bring in upper plane singularities and consisted of the complex conjugate singularity structures in the complex plane. Therefore, upper panel of the Hilbert transform and the spectral representation is completely equivalent for propagators.

Analytical model

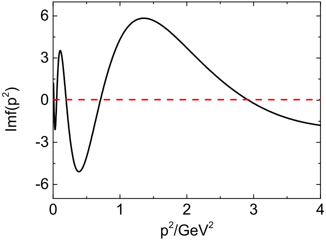
Firstly, we try an analytical model with singularities only on the lower plane,

$$\frac{0.2}{p^2+0.1}+\frac{0.8}{p^2+2+i}$$



Analytical model

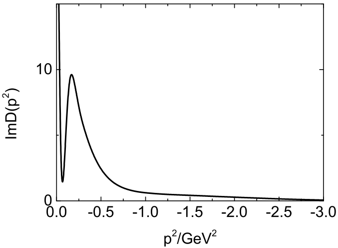
Reconstructing propagator from function $\frac{0.5}{p^2+1+i} + \frac{0.5}{p^2+1-i}$. .



- Direct continuation is $Imf(-p^2) = 0$.
- The reconstructed spectral from Hilbert transform is severely oscillated, because there is singularity on the upper plane

Ghost propagator

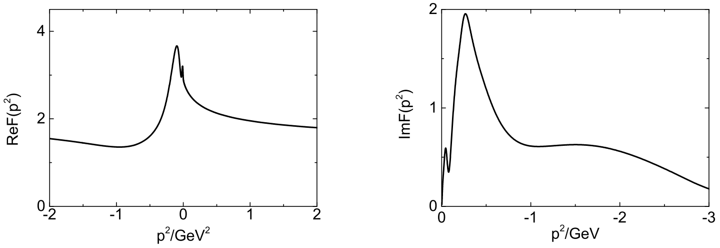
Now we can go to the realistic case. We first study the ghost propagator defined as $\frac{F(p^2)}{p^2}$. The Euclidean propagator is computed through ghost DSE with a massive gluon model:$D(k^2) = \frac{1}{k^2+\frac{m^4}{k^2+m^2}}$.



The spectral function of ghost propagator is positive-definite, which indicates that there're no singularities on the upper plane.

Ghost dressing function

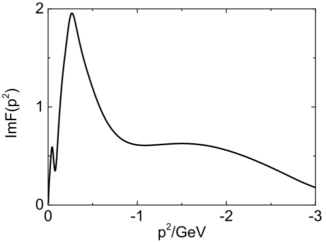
The dressing function $F(p^2)$ for both real part and imaginary part:

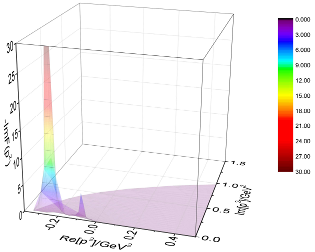


- The real part monotonously decreases in Euclidean space.
- The imaginary part in Minkowski space are mostly consisted of two peaks.

Ghost dressing function

We then compare the imaginary part of dressing function $F(p^2)$ in Minkowski space with the results from direct complex plane computation with Parabola contour.

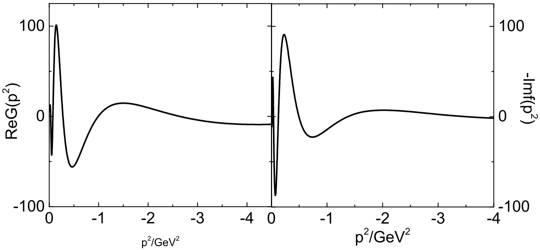




- Both consisted of two peaks at $p^2 = -0.05$ GeV^2 and $p^2 = -0.28$ GeV^2.
- No singularities on upper plane in the region that the Parabola contour can reach to.
- The Hilbert transform can obtain the information of the entire domain on time-like.

Gluon propagator

If we look into the gluon propagator, the behaviour is completely different. Both the real and the imaginary part of gluon propagator are oscillated severely. It indicates that the spectral function representation for gluon propagator is completely sabotaged by the complex conjugate singularities.



Summary

Summary:

- Developing an improved framework to compute meson mass and decay constant beyond Rainbow-ladder approxiamation
- Computing the properties of heavy-light mesons, including D mesons and B mesons, and both vector mesons and pseudoscalar mesons.
- Introducing a novel method to obtain the complex information of propagators.