Gluon propagator and vertex: from lattice QCD to instantons

F. de Soto

In collaboration with A. Athenodorou, Ph. Boucaud, J. Rodríguez-Quintero, S. Zafeiropoulos

Pablo de Olavide University (Seville, Spain)

fcsotbor@upo.es

NPQCD18 @ UPO Sevilla, 9/11/2018

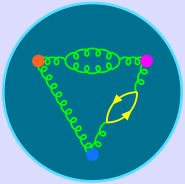
Motivation

QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = \sum_{q=u,d,\cdots} \underbrace{\overline{q}(i\gamma^\mu D_\mu - m_q)q}_{quarks} - \underbrace{\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a}_{\text{gluons}}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_0 f^a{}_{bc} A^b_\mu A^c_\nu$$

$$D_\mu = \partial_\mu + i g_0 A^a_\mu \frac{\lambda_a}{2}$$



Perturbative methods not useful for low-energy QCD: confinement and chiral symmetry breaking.

Non-perturbatice techniques and phenomenologic approaches:

- Lattice QCD
- Dyson-Schwinger Equations
- Operator Product Expansion
- SVZ sum rules
- Semiclassical aproximations

Motivation

Classical solutions of $\mathcal{L}_{\rm Y.M.}$

$$S = \frac{1}{2}\int d^4x \; G_{\mu\nu}^2 \geqslant 0 \; ; \qquad \frac{\delta S}{\delta G_\nu} = D_\mu G_{\mu\nu} = 0 \qquad \rightarrow \qquad A_I$$

- U(1) axial anomaly

$$\partial_\mu J_{5,\mu}(x) = \frac{g^2 N_F}{8\pi^2} Tr(G_{\mu\nu} \tilde{G}_{\mu\nu})$$

- Dirac operator zero modes

$$n_R(\psi_0); \; n_L(\gamma_5\psi_0)$$

- Topological charge

$$Q = n_L - n_R$$

Classical solutions of $\mathcal{L}_{\mathrm{Y.M.}}$

$$S = \frac{1}{2}\int d^4x\ G^2_{\mu\nu} \geqslant 0\ ; \qquad \frac{\delta S}{\delta G_\nu} = D_\mu G_{\mu\nu} = 0 \qquad \rightarrow \qquad A_I$$

- U(1) axial anomaly

$$\partial_\mu J_{5,\mu}(x) = \frac{g^2 N_F}{8\pi^2} Tr(G_{\mu\nu}\tilde{G}_{\mu\nu})$$

- Dirac operator zero modes

$$n_R(\psi_0);\ n_L(\gamma_5\psi_0)$$

- Topological charge

$$Q = n_L - n_R$$

- Genuinely non-perturbative
- QCD vacuum
- Condensates
- SBχS?
- Deep inelastic scattering?
- Mass generation?
- Confinement?
- Gluon Green functions?

[Rev. Mod. Phys. 70, (1998) 323]

Motivation

Instanton gauge field

Only exact solution [PLB59 (1975) 85]:

$$A_I \equiv A^a_\mu(x) = 2\bar{\eta}^a_{\mu\nu}\frac{x_\nu}{x^2}\phi\left(\frac{x}{\rho}\right)$$

with

$$\phi_{BPST}(z) = \frac{1}{1+z^2}$$

- Multi-instanton solutions are constructed by the sum of those solutions.

$$A = \sum A_I$$

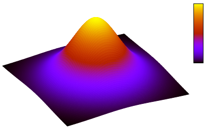
- The profile ϕ is modified at large distances due to instanton *interactions* [NPB245 (1984) 259].

Finite action

$$S_I = \int d^4x S(x) = \frac{8\pi^2}{g^2}$$

topological

$$Q = \int \partial_\mu K_\mu = \pm 1$$



Computing $\alpha_{\rm MOM}$ from the lattice

I Landau gauge fixing $\partial_\mu A_\mu = 0$

II Gluon propagator

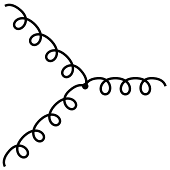
$$\langle \widetilde{A}^a_\mu(k)\widetilde{A}^a_\mu(-k)\rangle \to G^{(2)}(k^2)$$

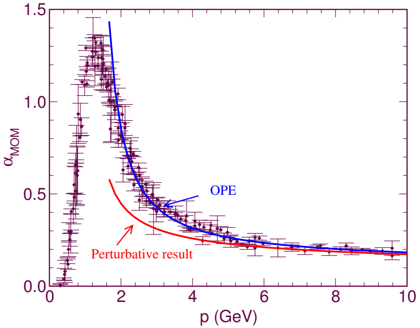
III Three gluon vertex

$$\langle \widetilde{A}^a_\mu(k_1)\widetilde{A}^b_\nu(k_2)\widetilde{A}^b_\rho(k_3)\rangle_{k_1^2=k_2^2=k_3^2} \to G^{(3)}(k^2)$$

IV The coupling is defined by the ratio:

$$\alpha_{\rm MOM}(k^2) = \frac{1}{4\pi}\frac{(k^6 G^{(3)}(k^2))^2}{(k^2 G^{(2)}(k^2))^3}$$

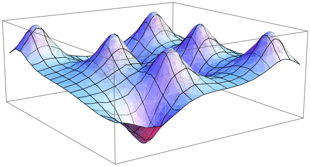




Gluon Green functions in an instanton background

- Independent pseudo-particle sum ansantz:

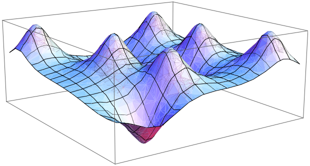
$$A^a_\mu(x) = \sum_i 2R_i^{a\alpha}\bar{\eta}^a_{\mu\nu}\frac{x_\nu - z^i_\nu}{(x-z^i)^2}\phi\left(\frac{x-z^i}{\rho_i}\right)$$



in $\langle A^2\rangle_{min}$ Landau gauge.

I Independent pseudo-particle sum ansantz:

$$A^a_\mu(x) = \sum_i 2R_i^{a\alpha}\bar{\eta}^a_{\mu\nu}\frac{x_\nu - z^i_\nu}{(x-z^i)^2}\phi\left(\frac{x-z^i}{\rho_i}\right)$$



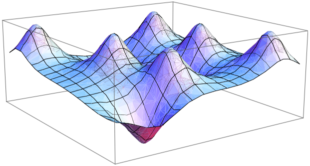
in $\langle A^2\rangle_{min}$ Landau gauge.

II Fourier transform

$$\mathbf{A}^{\mathbf{a}}_\mu(\mathbf{x})[\phi(z)] \rightarrow \widetilde{\mathbf{A}}^{\mathbf{a}}_\mu(\mathbf{k})[I(k\rho)] \; ; \qquad I(s) = \frac{8\pi^2}{s}\int_0^\infty dz\; z\; J_2(sz)\;\phi(z)$$

I Independent pseudo-particle sum ansantz:

$$A^a_\mu(x) = \sum_i 2R_i^{a\alpha}\bar{\eta}^a_{\mu\nu}\frac{x_\nu - z^i_\nu}{(x-z^i)^2}\phi\left(\frac{x-z^i}{\rho_i}\right)$$



in $\langle A^2\rangle_{min}$ Landau gauge.

II Fourier transform

$$\mathbf{A}^{\mathbf{a}}_\mu(\mathbf{x})[\phi(z)] \to \widetilde{\mathbf{A}}^{\mathbf{a}}_\mu(\mathbf{k})[I(k\rho)] \; ; \qquad I(s) = \frac{8\pi^2}{s}\int_0^\infty dz\; z\; J_2(sz)\;\phi(z)$$

III Neglecting instanton correlations [PRD70, (2004) 114503]

$$\mathbf{G}^{(\mathbf{m})}(\mathbf{k}^2) \sim \mathbf{n}\; \mathbf{k}^{2-\mathbf{m}}\langle\rho^{\mathbf{3m}}\mathbf{I}^{\mathbf{m}}(\mathbf{k}\rho)\rangle$$

with $\langle\cdots\rangle$ the average over instanton sizes.

The ratio of Green functions defining $\alpha_{\rm MOM}(k)$ is then:

$$\alpha_{\rm MOM}(k^2) = \frac{1}{4\pi} \frac{\left(k^6 G^{(3)}(k^2)\right)^2}{\left(k^2 G^{(2)}(k^2)\right)^3}$$

α_{MOM} in an instanton background

The ratio of Green functions defining $\alpha_{\mathrm{MOM}}(k)$ is then:

$$\alpha_{\mathrm{MOM}}(k^2) = \frac{1}{4\pi} \frac{(k^6 G^{(3)}(k^2))^2}{(k^2 G^{(2)}(k^2))^3} = \frac{k^4}{18\pi n} \frac{\langle \rho^9 I^3(k\rho) \rangle^2}{\langle \rho^6 I^2(k\rho) \rangle^3}$$

The ratio of Green functions defining $\alpha_{\mathrm{MOM}}(k)$ is then:

$$\alpha_{\mathrm{MOM}}(k^2) = \frac{1}{4\pi}\frac{(k^6 G^{(3)}(k^2))^2}{(k^2 G^{(2)}(k^2))^3} = \frac{k^4}{18\pi n}\frac{\langle \rho^9 I^3(k\rho)\rangle^2}{\langle \rho^6 I^2(k\rho)\rangle^3} = \frac{k^4}{18\pi n} \times \begin{cases} 1 + 48\frac{\delta\rho^2}{\langle\rho\rangle^2} & k \ll \rho^{-1} \\ 1 & k \gg \rho^{-1} \end{cases}$$

$\alpha_{\rm MOM}$ in an instanton background

The ratio of Green functions defining $\alpha_{\rm MOM}(k)$ is then:

$$\alpha_{\rm MOM}(k^2) = \frac{1}{4\pi}\frac{(k^6 G^{(3)}(k^2))^2}{(k^2 G^{(2)}(k^2))^3} = \frac{k^4}{18\pi n}\frac{\langle \rho^9 I^3(k\rho)\rangle^2}{\langle \rho^6 I^2(k\rho)\rangle^3} = \frac{k^4}{18\pi n} \times \begin{cases} 1 + 48\frac{\delta\rho^2}{\langle\rho\rangle^2} & k \ll \rho^{-1} \\ 1 & k \gg \rho^{-1} \end{cases}$$

Instanton detector

In an instanton background $\alpha_{\rm MOM}(k)$ should exhibit a $\sim k^4$ behavior sensitive to the instanton density n [JHEP 04 (2003) 005].

- This result is independent of the instanton profile function ϕ / I
- For small k depends on the width of the instanton distribution, $\delta\rho/\rho$

Lattice setup's for $N_f = 0$:

- Tree-level Symanzik

β	a	V	confs.
3.8	0.285fm	$(13.7\text{fm})^4$	1050
3.9	0.243fm	$(15.6\text{fm})^4$	2000
4.2	0.141fm	$(4.5\text{fm})^4$	420

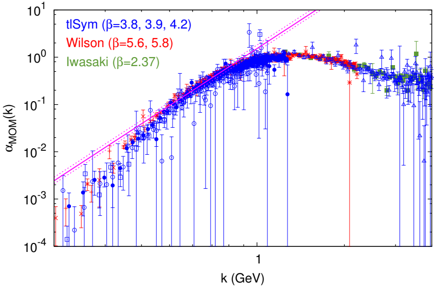
- Wilson

5.8	0.140fm	$(6.72\text{fm})^4$	960
5.6	0.235fm	$(11.3\text{fm})^4$	1920
5.6	0.235fm	$(12.3\text{fm})^4$	1790

- Iwasaki

2.37	0.140fm	$2.8^3 \times 5.6\text{fm}^4$

Details for the lattice setup in: **[PLB 760 (2016) 354]**



Instanton prediction $\alpha_{\rm MOM}(k) \approx ck^4/18\pi n$ works for $k \in (0.3, 0.9)\text{GeV}^{-1}$

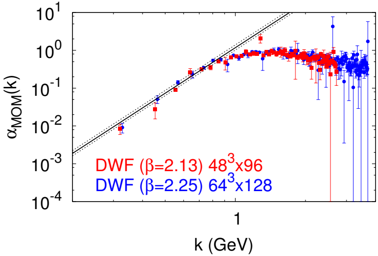
$$n \approx 12(1)\text{fm}^{-4}$$

(Assuming $c \approx 1.6$, from $\frac{\delta\rho^2}{\langle\rho\rangle^2} \approx 0.014$ [PRD58 (1998) 014505])

This method can be applied to compute instanton density without applying any smoothing technique, and also for $N_F \neq 0$ (JHEP 02 (2018) 140)

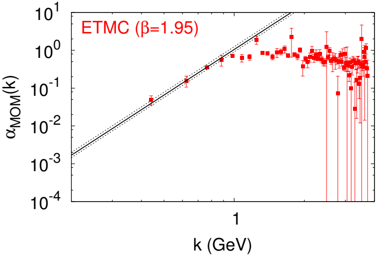
This method can be applied to compute instanton density without applying any smoothing technique, and also for $N_F \neq 0$ (JHEP 02 (2018) 140)

DWF ($N_F = 2+1$, RBC & UKQCD)



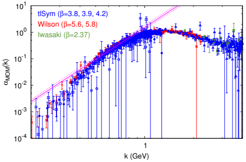
$$\frac{n_{N_F=2+1}}{n_{N_F=0}} = 1.3(1)$$

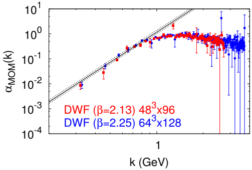
tmQCD ($N_F = 2+1+1$, ETMC)

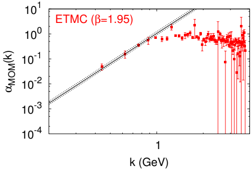


$$\frac{n_{N_F=2+1+1}}{n_{N_F=0}} = 1.5(1)$$

Instanton density from IR running: *resumé*

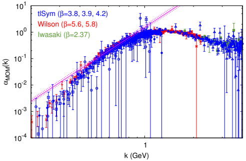


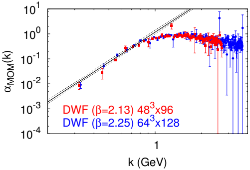


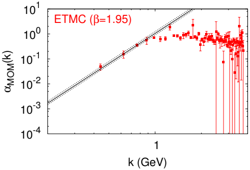


- Instanton k^4 law works for $0.3GeV \lesssim k \lesssim 1GeV$
- Rather large instanton density $\sim 10\mathrm{fm}^{-4}$
- $\sim$ 40% density increase due to dynamical quarks

Instanton density from IR running: *resumé*







- Instanton k^4 law works for $0.3 GeV \lesssim k \lesssim 1 GeV$
- Rather large instanton density $\sim 10\mathrm{fm}^{-4}$
- $\sim 40\%$ density increase due to dynamical quarks
- What if we eliminated quantum fluctuations?

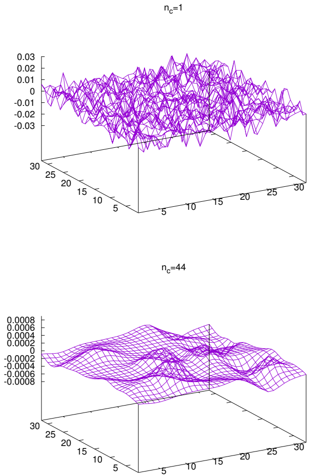
Instantons have been traditionally studied from lattice QCD via the removal of short-range (UV) fluctuations...

UV removal

Different methods have been used:

- Cooling
- Gradient flow
- APE spearing
- HYP smearing
- STOUT smearing

Topological charge density (2D cut)



Cooling, Wilson Flow and smearing techniques

Wilson Flow

- Flow Equation:

$$\partial_\tau B_\mu(\tau,x) = D_\nu G_{\mu\nu}(\tau,x) \ ,$$

  with $B_\mu(0,x) = A_\mu(x)$

- The flown fields are given by:

$$B_\mu(\tau,x) = \int d^4y \, \frac{e^{-\frac{(x-y)^2}{4\tau}}}{(4\pi\tau)^2} A_\mu(x)$$

- Has a smoothing effect of radius $\sqrt{8\tau}$ [JHEP 08 (2010) 071]

- Flow time usually expressed in terms of t_0 defined by $\sqrt{8t_0} = 0.3fm$

Cooling

- Updates gauge fields U by U^{new} minimizing the action.

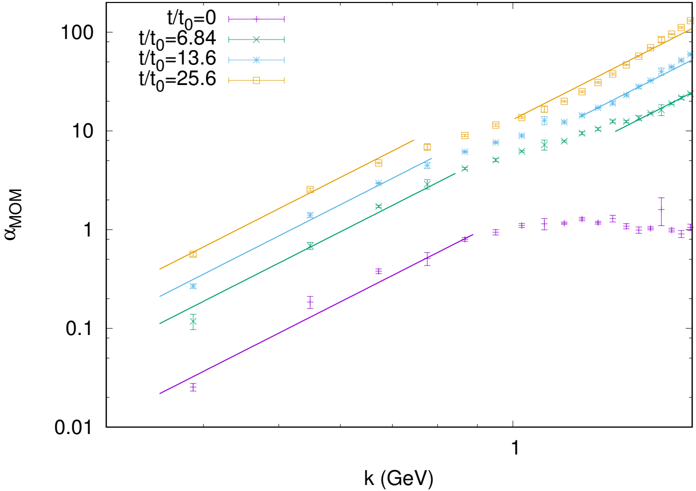
Smearing

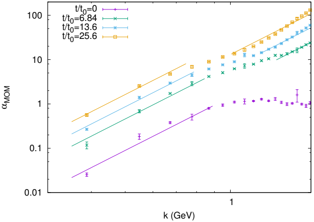
- The different types of smearing create *fat links* from the original ones and surrounding staples.

They provide similar topological properties [C. Alexandrou *et al.* PRD92, 125014 (2015) & arXiv:1708.00696]

$\alpha_{\mathrm{MOM}}(k)$ after Wilson Flow

Asymptotic freedom disappears and k^4-law emerges [PLB760 (2016) 354]





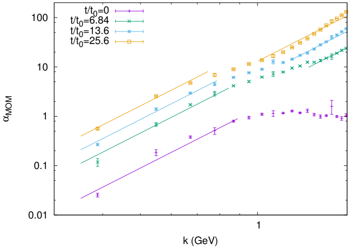
- Different slopes at small and large k are associated to the instanton distribution.

$$\alpha_{\mathrm{MOM}}(k^2) = \frac{k^4}{18\pi n} \times \begin{cases} 1 + 48\frac{\delta\rho^2}{\langle\rho\rangle^2} & k \ll \rho^{-1} \\ 1 & k \gg \rho^{-1} \end{cases}$$

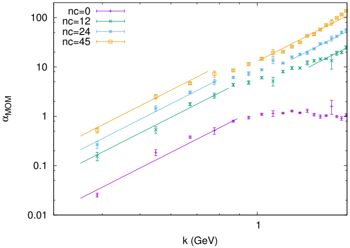
- At large momenta we can obtain instanton density n.
- At small momenta the slope is a factor $c = 1 + 48\frac{\delta\rho^2}{\langle\rho\rangle^2}$ times larger.
- If $\frac{\delta\rho^2}{\langle\rho\rangle^2} \approx 0.014$ [PRD58 (1998) 014505], $c \approx 1.6$ and $n = 12(1)\mathrm{fm}^{-4}$

Comparison with cooling and smearing techniques

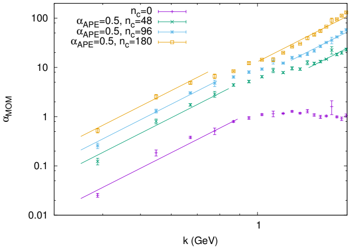
Wilson Flow



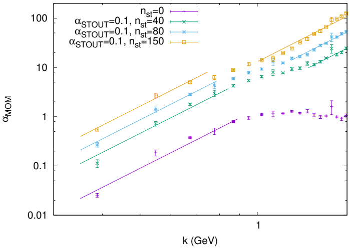
Cooling



APE smearing

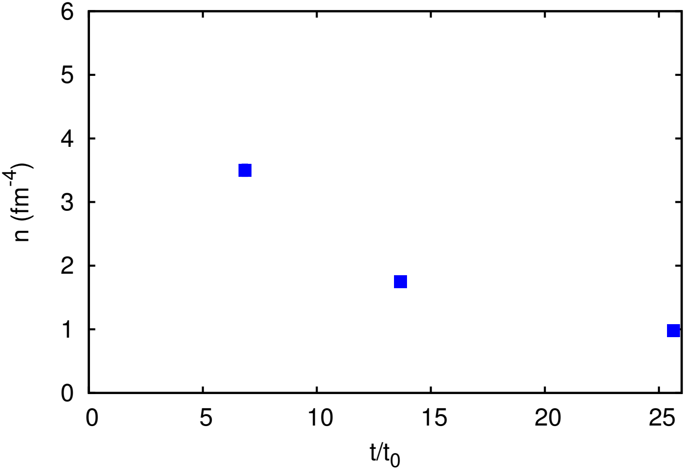


STOUT smearing



Instanton density strongly depends on the smoothing

The different slopes correspond to different instanton densities, i.e., the density decreases with the smoothing...



The same is obtained with equivalent cooling and APE & STOUT smearing

α_{MOM} and individual Green functions...

If the profile $\phi(x)$ is a BPST one (or when $k \gg \rho^{-1}$):

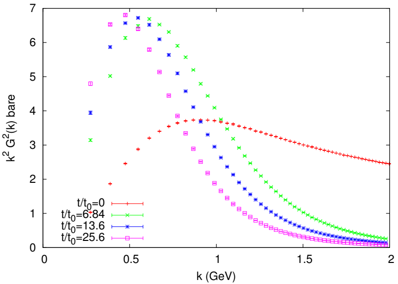
$$\mathbf{G}^{(\mathbf{m})}(\mathbf{k}^2) \sim \mathbf{n}\ \mathbf{k}^{2-\mathbf{m}} \langle \rho^{3\mathbf{m}} \mathbf{I}^{\mathbf{m}}(\mathbf{k}\rho) \rangle$$

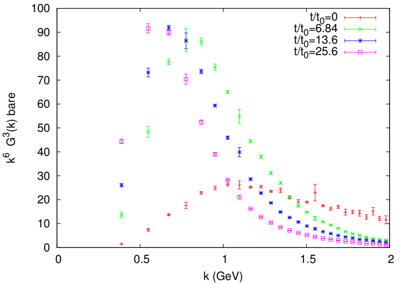
If the profile $\phi(x)$ is a BPST one (or when $k \gg \rho^{-1}$):

$$\mathbf{G^{(m)}(k^2) \sim n\, k^{2-m} \langle \rho^{3m} I^m(k\rho) \rangle} \quad \sim \quad \begin{cases} k^2 G^{(2)}(k^2) \sim n k^{-4} \\ k^6 G^{(3)}(k^2) \sim n k^{-4} \end{cases}$$

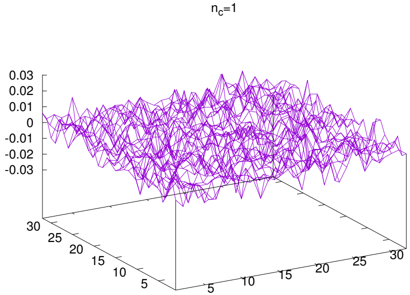
[Phys. Rev. D70 (2004) 114503]

...





Wilson flow / cooling / smearing eliminate short-distance fluctuations



Topological charge density

For BPST instantons,

$$Q(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}G_{\mu\nu}G_{\mu\nu}$$

behave as:

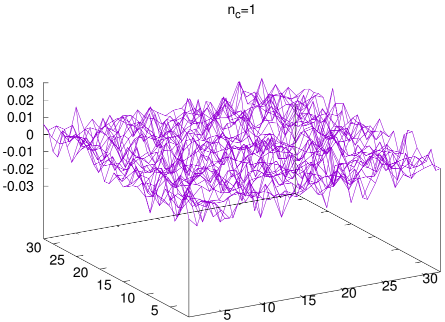
$$Q(x) = \frac{6}{\pi^2\rho^4}\left(\frac{\rho^2}{x^2+\rho^2}\right)^4$$

For modified profiles this shape

remains unchanged for small x

Locating instantons after smoothing

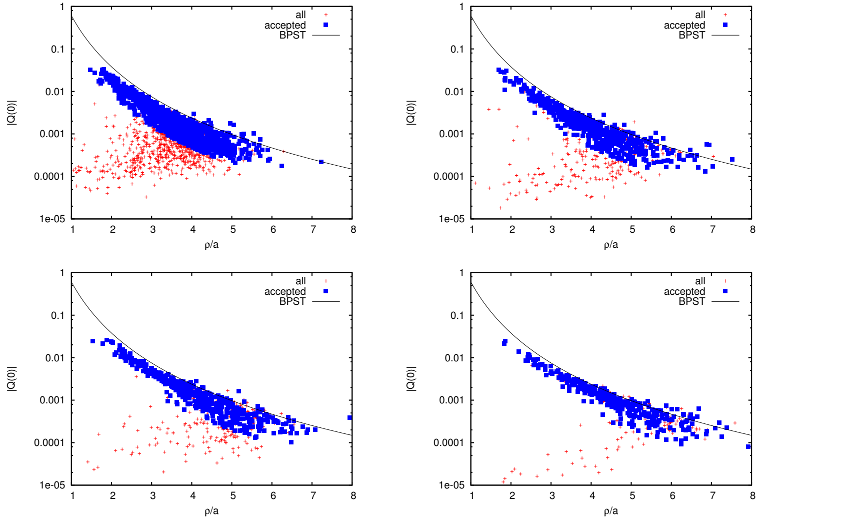
Wilson flow / cooling / smearing eliminate short-distance fluctuations



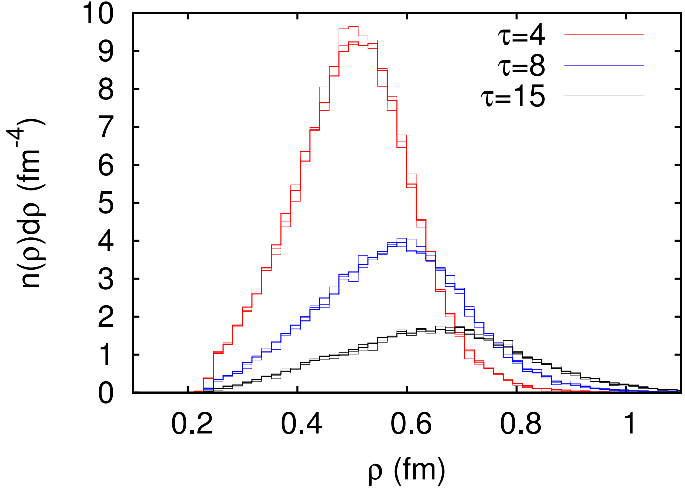
- Search for local extremes $x / Q(x) > Q(x \pm \hat{\mu})$
- Fit $Q(x \pm \hat{\mu})$ to BPST (like PRD 88 (2013) 034501)
- Check self-duality
- Discard close pairs when $r_{ij}^2 < \rho_i \rho_j$

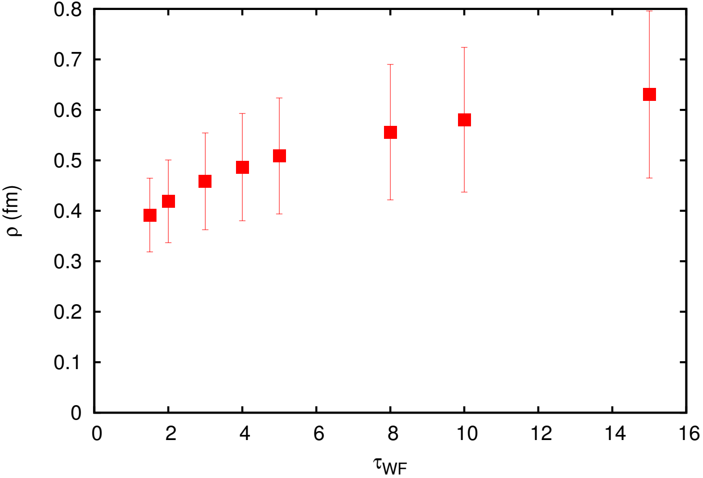
Results (instanton localization)

tlSym 32^4, $\beta = 4.2$ at $\tau = 4$, 8, 10 and 15

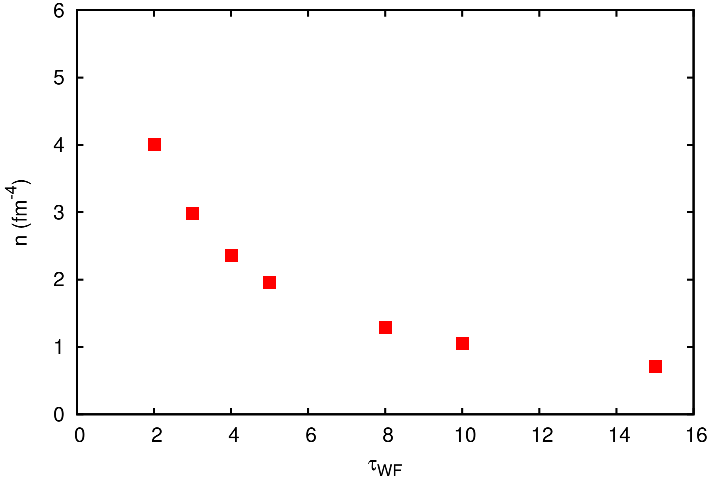


Instanton sizes (tlSym 32^4, $\beta = 4.2$)



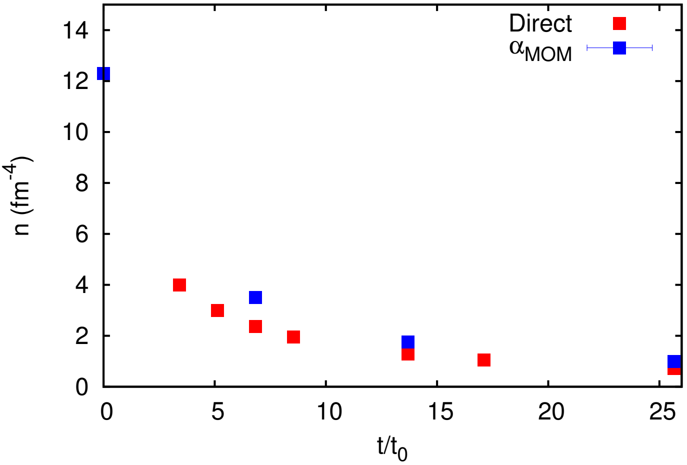

0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ρ (fm)
0
2
4
6
8
10
12
14
16
τ_{WF}

Instanton density (tlSym 32^4, $\beta = 4.2$)

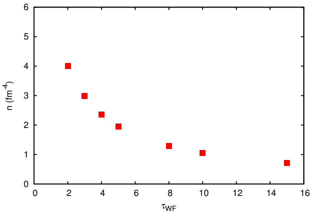


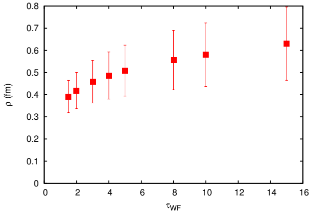
Direct counting vs $\alpha_{\rm MOM}(k^2)$ (tlSym 32^4, $\beta = 4.2$)

We can compare the densities extracted from direct detection and $\alpha_{\rm MOM}$:

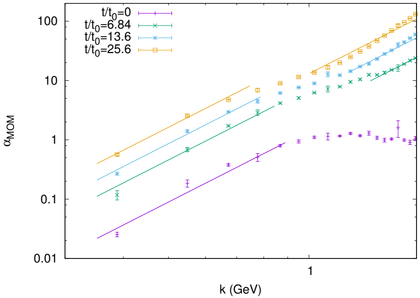


Direct counting vs $\alpha_{\mathrm{MOM}}(k^2)$ (tlSym 32^4, $\beta = 4.2$)





The evolution of the Instanton Liquid parameters can be observed both from $\alpha_{\mathrm{MOM}}(k)$ and the direct localisation.



Instanton evolution with smoothing

Annihilation model

If instanton annihilation is a first order process:

$$\frac{dn_I}{d\tau} = \frac{dn_A}{d\tau} = -\lambda n_I n_A$$

and

$n = n_I + n_A \gg q = n_I - n_A$,

$$\frac{dn}{d\tau} \approx -\frac{\lambda}{2} n^2$$

whose solution for $\lambda = cte$ is

$$\frac{1}{n(\tau)} = \frac{1}{n(0)} + \frac{1}{2}\lambda\tau$$

Annihilation model

If instanton annihilation is a first order process:

$$\frac{dn_I}{d\tau} = \frac{dn_A}{d\tau} = -\lambda n_I n_A$$

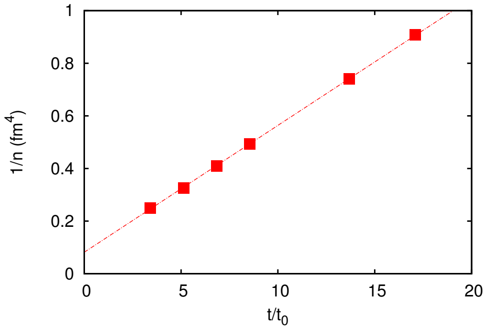
and

$n = n_I + n_A \gg q = n_I - n_A$,

$$\frac{dn}{d\tau} \approx -\frac{\lambda}{2} n^2$$

whose solution for $\lambda = cte$ is

$$\frac{1}{n(\tau)} = \frac{1}{n(0)} + \frac{1}{2}\lambda\tau$$



$n \sim 12\mathrm{fm}^{-4}$

Conclusions

- Instantons explain the running of α_{MOM} in the IR
- Instanton density increases with N_F
- After smoothing, instantons dominate also the running at large momenta (Λ_{QCD} dissapeared!)
- Instanton density n and size ρ evolve with the smoothing
- A simple model for IA annihilation allows to understand the evolution.